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Perturbation from an elliptic Hamiltonian of degree four—IV figure eight-loop

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Abstract

The paper deals with Liénard equations of the form $\dot{x} = y$, $\dot{y} = P(x) + yQ(x)$ with P and Q polynomials of degree, respectively, 3 and 2. Attention goes to perturbations of the Hamiltonian vector fields with an elliptic Hamiltonian of degree four, exhibiting a figure eight-loop. It is proved that the least upper bound of the number of zeros of the related elliptic integral is five, and this is a sharp bound, multiplicity taken into account. Moreover, if restricting to the level curves "inside" a saddle loop or "outside" the figure eight-loop the sharp upper bound is respectively two or four; also the multiplicity of the zeros is at most four. This is the last one in a series of papers on this subject. The results of this paper, together with (J. Differential Equations 176 (2001) 114; J. Differential Equations 175 (2001) 209; J. Differential Equations, to be published), largely finish the study of the cubic perturbations of the elliptic Hamiltonians of degree four and presumably provide a complete description of the number and the possible configurations of limit cycles for cubic Liénard equations with small quadratic damping. As a special case, we obtain a configuration of four limit cycles surrounding three singularities together with a "small" limit cycle which surrounds one of the singularities.

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1. Introduction

This is the last paper in a series of four, the others being [DL1,DL2,DL3]. In [DL1] we gave an extensive introduction to the subject and motivated why we are

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interested in it. Let us recall that our aim is to make a precise calculation of the number and relative position of the zeros of the Abelian integrals that have to be dealt with in order to study the perturbations of vector fields, having an elliptic Hamiltonian of degree four, inside the space of Liénard equations of type (3,2). This means that we deal with Hamiltonians of the form

$$H(x,y) = \frac{1}{2}y^2 + \frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2 \tag{1}$$

with $a \neq 0$.

If the level curves of $H(x,y)$ contain compact components, then there are five different types, shown in Figs. 1 (A)–(E); they are, respectively, called the cases of two saddle cycle, saddle loop, global centre, cuspidal loop and figure eight-loop. Note that case (A) is a limiting case of (B); and case (D) is a limiting case of (C) and (E).

Let us denote by N_L, $L = A, B, \ldots, E$, the sharp upper bounds of the number of zeros, counting the multiplicity, of the elliptic integrals obtained by integrating the 1-forms $y(\alpha + \beta x + x^2)\,dx$ over the compact level curves of the Hamiltonian $H(x,y)$ of case (L) in Fig. 1, where α and β are arbitrary constants. Note that the trivial zero, corresponding to the centre, is not included in N_L.

In this series of four papers we not only want to make a precise calculation of N_L in the five different cases, but also aim at describing, in as much detail as possible, the relative positions of these zeros and the related bifurcation set when changing the parameters a, b, c, α, β. In [DL1] we made a complete study of the case (B), linking it to the known results in case (A) (see [H]); we found that $N_B = 2$. In [DL2] we made a complete study of the interesting limiting case (D), proving among other results that $N_D = 4$. Moreover, if restricting to the level curves "inside" and "outside" the cuspidal loop, we found the sharp upper bound to be, respectively, 2 and 3.

In [DL3] we found that $N_C = 4$ and we proved that the 4 zeros of the elliptic integral can be simple or multiple exhibiting a complete unfolding of a zero of multiplicity four. We would give a complete list of all possibilities and bifurcations concerning the position of these zeros. In the present paper we want to study the

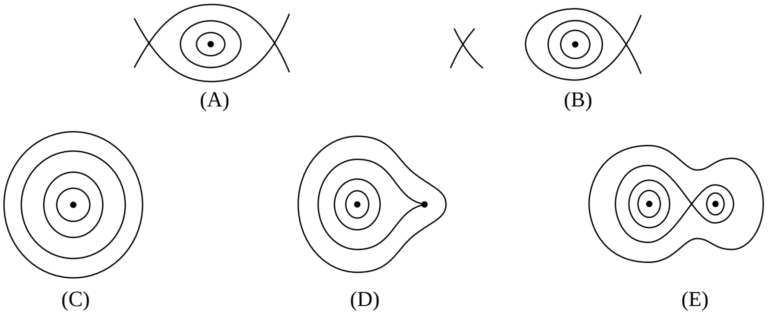

Fig. 1. The level curves of $H(x,y)$.

remaining case (E). We intend to prove that $N_E = 5$. In fact there are three kinds of zeros for the elliptic integrals in the case (E), depending on whether we integrate over compact level curves "inside" the left loop, inside the right loop, or outside the figure eight-loop; let us denote their respective number by $((n_1, n_2), n_3)$, depending on (α, β). We are going to prove that $n_3 \leqslant 4$, $n_1 + n_2 \leqslant 2$ and $n_1 + n_2 + n_3 \leqslant 5$. The complete list of possibilities on $((n_1, n_2), n_3)$ is given in Theorem 1.1.

Before stating Theorem 1.1 we will first simplify the equations under study. As mentioned above the Hamiltonians under consideration have form (1) and their level curves are shown in Fig. 1(E). Without loss of generality we can put the saddle point and the two centres of the corresponding Hamiltonian vector fields at, respectively, $(0,0)$, $(-1,0)$ and $(\lambda, 0)$, where $0 < \lambda \leqslant 1$. The systems that we want to study have the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x(x+1)(x-\lambda) + \delta(\alpha + \beta x + x^2)y, \end{cases} \tag{2_δ}$$

where α and β are arbitrary constants and δ is small. We have hence changed the Hamiltonians (1) to the simpler form

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 + \frac{1-\lambda}{3}x^3 - \frac{\lambda}{2}x^2 \tag{3}$$

with a unique parameter $\lambda \in (0,1]$. If $\lambda \to 0$, then (3) tends to the cuspidal loop case, treated in [DL2]. In Fig. 2 we represent the bifurcation diagram of the zeros, recalling that (k, ℓ) stands for k zeros (resp. ℓ zeros) related to level curves inside (resp. outside) the cuspidal loop. In this figure C stands for a zero at the cuspidal loop, H for a zero at the antisaddle, and the other curves represent double zeros; there is one triple zero.

If $\lambda = 1$, then we get a symmetric case which was studied, e.g. in [CLW,GH]. In the latter case the possible configurations of zeros, and the related bifurcation diagram with respect to (α, β), are represented in Fig. 3. H_L (resp. H_R) stands for a zero at the left (resp. right) antisaddle, S_L (resp. S_R) stands for a zero at the left (resp. right) saddle loop, S_A (resp. S_B) stands for a zero that will give rise to a saddle connect on above (resp. below) the saddle; the other curves represent double zeros.

Hence in the rest of the paper we will restrict to $\lambda \in (0,1)$. In that case the Hamiltonian (3) has 3 critical values $h_1 < h_2 < 0$, where

$$h_1 = -\frac{1}{12}(2\lambda + 1), \quad h_2 = -\frac{1}{12}\lambda^3(\lambda + 2). \tag{4}$$

We denote the compact components of the level curves of H by respectively $\Gamma_h^1 = \{(x,y) \mid H(x,y) = h,\ h \in (h_1, 0),\ x < 0\}$, $\Gamma_h^2 = \{(x,y) \mid H(x,y) = h,\ h \in (h_2, 0),\ x > 0\}$ and $\Gamma_h^3 = \{(x,y) \mid H(x,y) = h,\ h \in [0, +\infty)\}$; then Γ_h^1 (resp. Γ_h^2) surrounds the centre $(-1,0)$ (respectively, $(\lambda, 0)$), Γ_0^3 is the figure eight-loop, and Γ_h^3 surrounds the

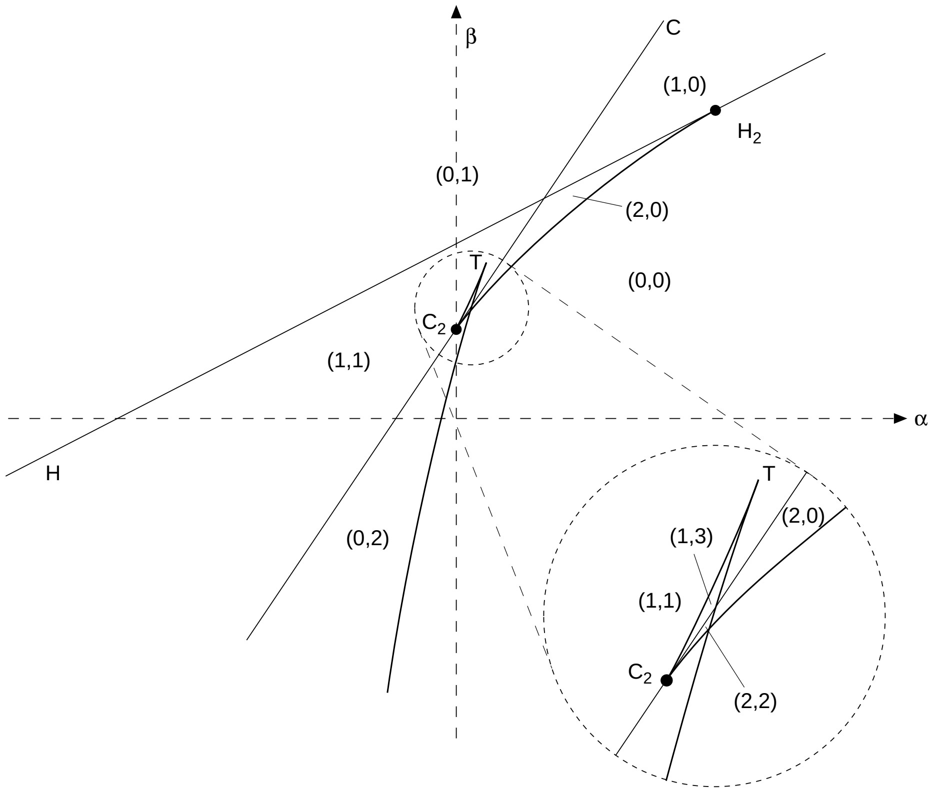

Fig. 2. Bifurcation diagram of zeros for $\lambda = 0$.

figure eight-loop. Later on we will also work with Γ_h where $\Gamma_h = \Gamma_h^3$ for $h \geqslant 0$ and $\Gamma_h = \Gamma_h^1 \cup \Gamma_h^2$ for $h < 0$.

Related to $(2)_\delta$, we consider the elliptic integrals

$$I^{(i)}(h) = \int_{\Gamma_h^i} (\alpha + \beta x + x^2) y \, dx = \alpha I_0^{(i)}(h) + \beta I_1^{(i)}(h) + I_2^{(i)}(h), \tag{5}$$

where

$$I_k^{(i)}(h) = \int_{\Gamma_h^i} x^k y \, dx. \tag{6}$$

We are now ready to state our main theorem.

Theorem 1.1. *Let us consider the Hamiltonians of form* (3) *and the related elliptic integrals as defined in* (5) *and* (6): *let n_i denote the number of zeros of $I^{(i)}(h)$ for $i = 1, 2, 3$, depending on (α, β).*

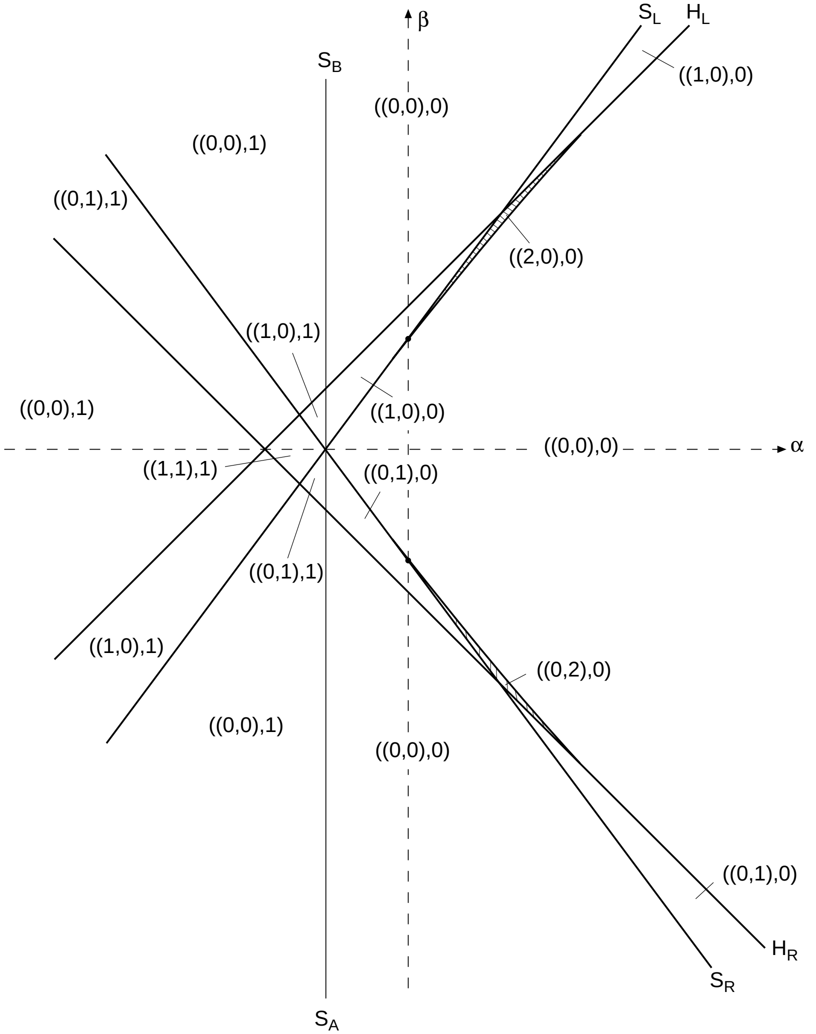

Fig. 3. Bifurcation diagram of zeros for $\lambda = 1$.

Following statements hold:

(1) *For all* $\lambda \in (0,1)$ *and for all* (α, β), $n_1 + n_2 \leqslant 2$, $n_3 \leqslant 4$, $n_1 + n_2 + n_3 \leqslant 5$.
(2) *For* $0 < \lambda \ll 1$ (*resp*. $0 < 1 - \lambda \ll 1$), *the possible configurations* $((n_1, n_2), n_3)$ *and the related bifurcation diagram are as represented in* Fig. 4 (*resp*. Fig. 5).
(3) *For all* $\lambda \in (0,1)$, *the possible configurations* (n_1, n_2) *and the related bifurcation diagram of the zeros of* $I^{(1)}$ *and* $I^{(2)}$ *are as represented in Fig*. 4 (*and Fig*. 5).
(4) *There exists a value* $\lambda^* \in (0,1)$, *where* $\lambda^* \approx 0.0549$, *such that for* $\lambda \in (0, \lambda^*)$ *the bifurcation diagram of the zeros of* $I^{(3)}(h)$ *is as represented in* Fig. 6(a), *and for*

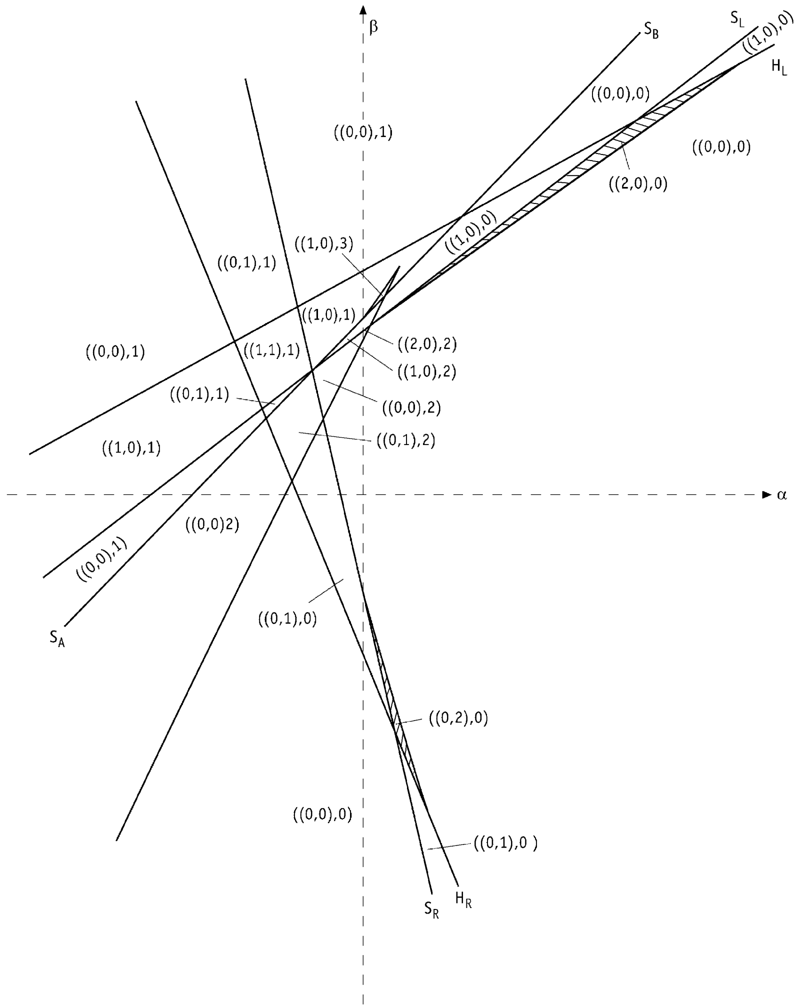

Fig. 4. Bifurcation diagram of zeros for $0 < \lambda \ll 1$.

$0 < \lambda - \lambda^* \ll 1$, *the bifurcation diagram of the zeros of* $I^{(3)}(h)$ *is as represented in Fig.* 6(b). *For* (α, β) *in the triangular region in which* $I^{(3)}(h)$ *has* 4 *zeros, the related* $(n_1, n_2) = (1, 0)$, *implying the existence of a* $((1,0),4)$-*configuration of zeros.*

(5) *There exists a value* $\lambda^\circ \in (\lambda^*, 1)$ *such that for* $0 < \lambda^\circ - \lambda \ll 1$ (*resp.* $0 < \lambda - \lambda^\circ \ll 1$) *the bifurcation diagram of the zeros of* $I^{(3)}(h)$ *is as represented in Fig.* 6(c) (*resp. Fig.* 6(d)).

In Figs. 4 and 5, the curves H_L, H_R, S_L, S_R, S_A and S_B represent the same situations as in Fig. 3; all the other curves represent double zeros, we have a unique triple zero in Fig. 4.

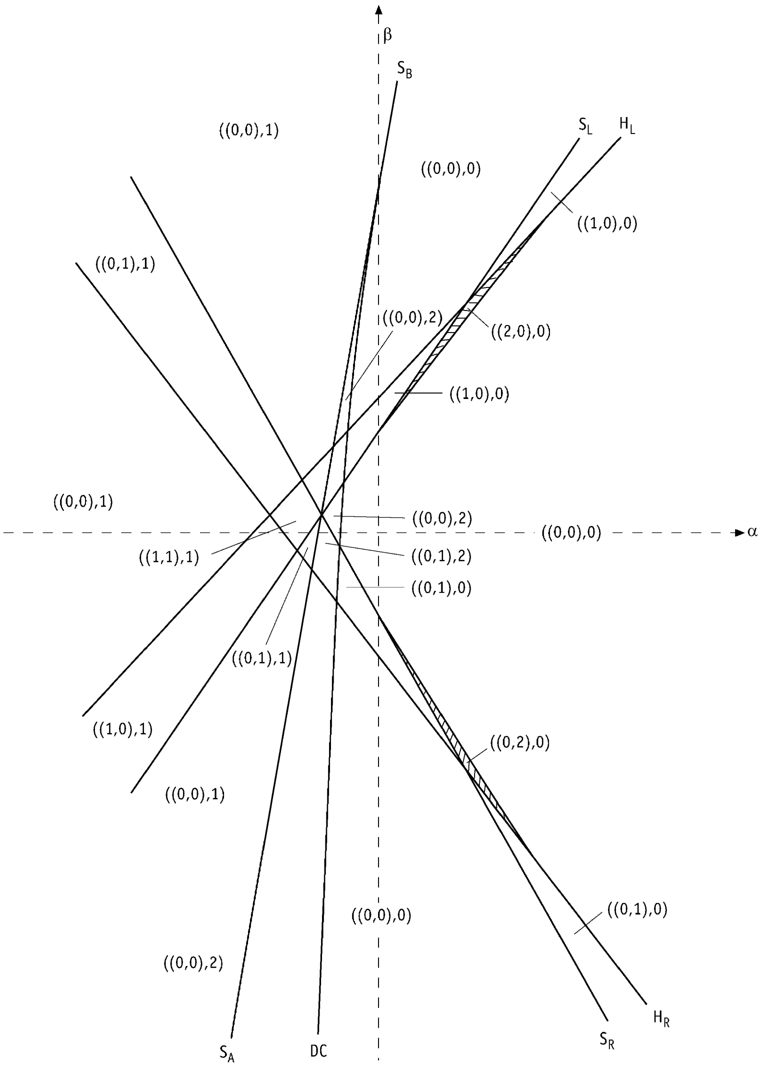

Fig. 5. Bifurcation diagram of zeros for $0<1-\lambda\ll 1$.

For $\lambda\downarrow 0$ the bifurcation diagram as represented in Fig. 4 tends to the limiting case $(\lambda=0)$ represented in Fig. 2. In fact the two lines $\overline{S_A\cup S_B}$ and S_L both tend to C, while the lines H_R and S_R both tend to the β-axis, where they do no longer coincide with a curve of bifurcation.

For $\lambda\uparrow 1$ the bifurcation diagram as represented in Fig. 5 tends to the limiting case $(\lambda=1)$ represented in Fig. 3. The line $\overline{S_A\cup S_B}$ and the curve DC both tend to the straight line $\overline{S_A\cup S_B}$ situated at $\{\alpha=-\frac{4}{5}\}$.

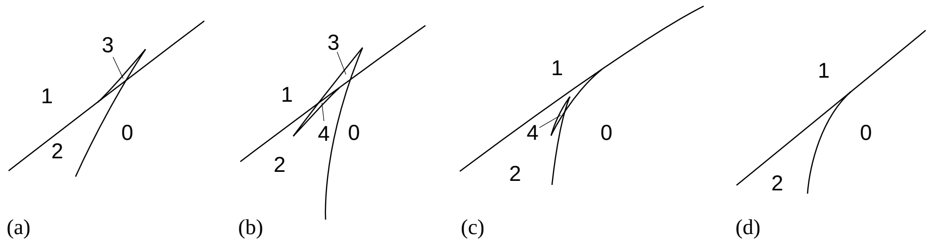

Fig. 6. Codimension-3 bifurcations for zeros of $I^{(3)}(h)$.

Theorem 1.1 contains the main results that we are able to prove. The value λ^* is unique, but we cannot yet prove λ° to be unique, although it probably is. It is also not yet possible to situate all the changes of the bifurcation diagram of the zeros of $I^{(3)}$ with respect to the bifurcation diagram of the zeros of $I^{(1)}$ and $I^{(2)}$. Of course with statement (1) we obtain sharp upperbounds on the precise number of zeros of $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$. Whether this result induces a similar one on the number of limit cycles for $(2)_\delta$, and $\delta \sim 0$, is not clear because of the presence of the figure eight-loop.

In any case the results described in (4) and (5) of Theorem 1.1, induce a similar result concerning the configuration of limit cycles of the corresponding systems $(2)_\delta$.

Recall that in [DL3] we obtained Liénard equations of type (3,2) with, respectively, a quadruple limit cycle or 4 simple limit cycles surrounding one focus. From the results in this paper we obtain Liénard equations of type (3,2) with, respectively, 4 simple limit cycles or a quadruple limit cycle surrounding three singularities. The first situation reveals also to occur in the presence of an extra ‘‘small’’ limit cycle, around one of the 3 singularities, giving a total of 5 limit cycles.

The proof of the theorem goes along the lines explained in [DL1]. In Section 2 we obtain a few simple preliminary results. In Section 3 we make a study in the (P, Q)-plane, where, as usual, $P(h) = \frac{I_1(h)}{I_0(h)}$, $Q(h) = \frac{I_2(h)}{I_0(h)}$; in fact we will need to consider $P_i(h) = \frac{I_1^{(i)}(h)}{I_0^{(i)}(h)}$ and $Q_i(h) = \frac{I_2^{(i)}(h)}{I_0^{(i)}(h)}$, with $i = 1, 2, 3$.

In Section 4 we make a study in the (h, ω)-plane, where, as in the previous papers, $\omega(h) = \frac{I_1''(h)}{I_0''(h)}$; in fact we again have to consider $\omega^{(i)}(h) = \frac{(I_1^{(i)})''(h)}{(I_0^{(i)})''(h)}$, and we also consider the related $v^{(i)}(h) = \frac{(I_2^{(i)})''(h)}{(I_0^{(i)})''(h)}$.

In Section 5 we make a study of the three curves in (h, P, Q)-space and in (h, P)-plane, and in Section 6 we combine all previous results to finish the proof of Theorem 1.1.

2. Preliminaries

We intend to study the elliptic integrals defined in (5) and (6), integrating over the compact level curves of the Hamiltonians (3). Since $I_0^{(i)}(h) > 0$ for $h > h_i$, $i = 1, 2$;

and $I_0^{(3)}(h) > 0$ for $h \geqslant 0$, we can define

$$P_i(h) = \frac{I_1^{(i)}(h)}{I_0^{(i)}(h)}, \quad Q_i(h) = \frac{I_2^{(i)}(h)}{I_0^{(i)}(h)}, \tag{7}$$

where $i = 1, 2, 3$. The definition is also valid for $h = h_i$, $i = 1, 2$, because of the following obvious conclusions.

Lemma 2.1. *As* $h \to h_1 + 0$, $P_1(h) \to -1$ *and* $Q_1(h) \to 1$; *as* $h \to h_2 + 0$, $P_2(h) \to \lambda$ *and* $Q_2(h) \to \lambda^2$.

A straightforward calculation leads to the following conclusion giving the values $P_3(0)$ and $Q_3(0)$.

Lemma 2.2. *For* $\lambda \in (0, 1)$

$$I_0^{(3)}(0) = \frac{4}{9}\sqrt{\lambda}(\lambda^2 + \lambda + 1) + \frac{2\sqrt{2}}{27}(1-\lambda)(\lambda+2)(2\lambda+1)\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}},$$

$$\begin{aligned} I_1^{(3)}(0) = & -\frac{\sqrt{\lambda}}{27}(1-\lambda)(10\lambda^2 + 19\lambda + 10) \\ & - \frac{\sqrt{2}}{162}(20\lambda^4 + 28\lambda^3 - 15\lambda^2 + 28\lambda + 20)\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}}, \end{aligned}$$

$$\begin{aligned} I_2^{(3)}(0) = & \frac{2\sqrt{\lambda}}{405}(70\lambda^4 + 65\lambda^3 - 54\lambda^2 + 65\lambda + 70) \\ & + \frac{\sqrt{2}}{243}(1-\lambda)(28\lambda^4 + 68\lambda^3 + 51\lambda^2 + 68\lambda + 28)\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}}. \end{aligned}$$

Taking $a = 1$, $b = 1 - \lambda$ and $c = -\lambda$ in (11) of [DL1], we obtain the following system of equations that hold for $P_i(h)$ and $Q_i(h)$, $i = 1, 2, 3$, on the corresponding intervals of h:

$$\begin{cases} \dot{h} = G(h), \\ \dot{P} = f(h, P, Q), \\ \dot{Q} = g(h, P, Q), \end{cases} \tag{8}$$

where

$$G(h) = 12^3 h(h-h_1)(h-h_2),$$

$$\begin{aligned} f(h,P,Q) = {} & 12(1-\lambda)h(12h - \lambda(\lambda+2)(2\lambda+1)) + (432h^2 + 12(4\lambda^4 - 9\lambda^3 - 17\lambda^2 \\ & - 9\lambda + 4)h - 12\lambda^3(\lambda+2)(2\lambda+1))P + 180(1-\lambda)(\lambda+1)^2 hQ \\ & + 2(1-\lambda)(12(\lambda^2 - 5\lambda + 1)h + \lambda^2(14\lambda^2 + 35\lambda + 14))P^2 \\ & + (180(\lambda^2+\lambda+1)h + 15\lambda^2(\lambda+2)(2\lambda+1))PQ, \end{aligned}$$

$$\begin{aligned} g(h,P,Q) = {} & -12h(12(\lambda^2+\lambda+1)h + \lambda^2(\lambda+2)(2\lambda+1)) + 24(1-\lambda)(12h + 7\lambda^3 \\ & + 13\lambda^2 + 7\lambda)hP + (864h^2 - 24(5\lambda^4 + 3\lambda^3 - 7\lambda^2 + 3\lambda + 5)h \\ & - 12\lambda^3(\lambda+2)(2\lambda+1))Q + 2(1-\lambda)(12(\lambda^2 - 5\lambda + 1)h + \lambda^2(14\lambda^2 \\ & + 35\lambda + 14))PQ + (180(\lambda^2+\lambda+1)h + 15\lambda^2(\lambda+2)(2\lambda+1))Q^2. \end{aligned}$$

3. The study in (P,Q)-plane

Lemma 3.1. (i) *In (P,Q)-plane the points $(P_i(0), Q_i(0))$, $i = 1,2,3$, are located on the same straight line*

$$L: 12\lambda + 14(\lambda-1)P - 15Q = 0, \tag{9}$$

(ii) $P_3(h) < \frac{\lambda-1}{3}$ *and* $\lim_{h\to+\infty} P_3(h) = \frac{\lambda-1}{3}$,
(iii) *For* $h \gg 1$, $Q_3(h) \sim a\sqrt{h}$, *where* $a > 0$ *is a constant.*

Proof. Conclusion (i) follows by taking $h = 0$ in the right sides of the last two equations of system (8), and by the fact that $(0, P_i(0), Q_i(0))$ is a singularity of (8) for $i = 1,2,3$.

Conclusions (ii) and (iii) can be obtained in the same way as for Lemma 2.8 of [DL3]. Note that the parameter λ here is not the same as in [DL3]. $\square$

Lemma 3.2. *For all* $\lambda \in (0,1)$ *we have*:

(i) $P_1'(h) > 0$, $Q_1'(h) < 0$ *for* $h \in (h_1, 0)$;
(ii) $P_2'(h) < 0$, $Q_2'(h) < 0$ *for* $h \in (h_2, 0)$;
(iii) $P_3'(h) > 0$ *for* $h > 0$.

Proof. The conclusions (i) and (ii) can be proved by using the monotonicity criterion for a ratio of two Abelian integrals as given in Example 5 of [LZ]. Conclusion (iii) can be proved by using the same method as for Lemma 2.2 of [DL3], based on a key idea borrowed from [L]. $\square$

Lemma 3.3. $\frac{Q_i'(h)}{P_i'(h)} \to \frac{Q_i(0)}{P_i(0)}$ *as* $h \to 0-0$ *for* $i = 1, 2$ *and* $h \to 0+0$ *for* $i = 3$.

The proof is exactly the same as for Lemma 11(i) of [DL2].

Lemma 3.4. $(\lambda - \lambda^*)\frac{d}{dh}\left(\frac{I_2^{(3)}(h)}{I_1^{(3)}(h)}\right)\Big|_{h=0} < 0$ *for* $\lambda \neq \lambda^*$, $\lambda \in (0,1)$, *where* $\lambda^* \approx 0.0549$ *is the unique solution of* $F(\lambda) = 0$ *for* $\lambda \in (0,1)$, *where*

$$F(\lambda) = B_0(\lambda) + B_1(\lambda)\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}} + B_2(\lambda)\left(\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}}\right)^2 \tag{10}$$

and

$$\begin{cases} B_0(\lambda) = \dfrac{4}{27}\lambda(\lambda-1)(10\lambda^2+19\lambda+10), \\ B_1(\lambda) = -\dfrac{2\sqrt{2\lambda}}{135}(20\lambda^4 - 49\lambda^2 + 20), \\ B_2(\lambda) = -\dfrac{8}{243}(\lambda-1)(\lambda+2)^2(2\lambda+1)^2. \end{cases}$$

Proof. $I_1^{(3)}(0)$ and $I_2^{(3)}(0)$ are given in Lemma 2.2. A calculation shows that

$$\frac{d}{dh}I_1^{(3)}(h)\Big|_{h=0} = -2\sqrt{2}\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}},$$

$$\frac{d}{dh}I_2^{(3)}(h)\Big|_{h=0} = 4\sqrt{\lambda} - \frac{4\sqrt{2}}{3}(\lambda-1)\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}}.$$

From these expressions we obtain

$$\frac{d}{dh}\left(\frac{I_2^{(3)}(h)}{I_1^{(3)}(h)}\right)\Big|_{h=0} = \frac{F(\lambda)}{(I_1^{(3)}(0))^2},$$

where $I_1^{(3)}(0) < 0$ and $F(\lambda)$ is given in (10). It is easy to check that $F(0+) > 0$ and $F(\lambda) < 0$ for $0 < 1 - \lambda \ll 1$. Hence to finish the proof of Lemma 3.4, it is enough to show that $F(\lambda) = 0$ has a unique solution λ^* for $\lambda \in (0,1)$.

We can write expression $F(\lambda)$ as $f(\lambda, A)$ if we write $A = \arctan \frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}} > 0$ for $\lambda \in (0,1)$. A straightforward calculation gives

$$F'(\lambda) = \frac{g(\lambda, A)}{1215\sqrt{\lambda}(\lambda+2)(2\lambda+1)},$$

where $g(\lambda, A) = C_0(\lambda) + C_1(\lambda)A + C_2(\lambda)A^2$, and

$$C_0(\lambda) = 18\sqrt{\lambda}(869\lambda^5 + 2600\lambda^4 + 1643\lambda^3 - 707\lambda^2 - 800\lambda - 140),$$

$$C_1(\lambda) = -15\sqrt{2}(\lambda+2)(2\lambda+1)(92\lambda^4 - 40\lambda^3 - 147\lambda^2 + 40\lambda + 28),$$

$$C_2(\lambda) = -40\sqrt{\lambda}(\lambda+2)^2(2\lambda+1)^2(10\lambda^2 + 7\lambda - 8).$$

If we now consider A as an independent variable, then we can express $F(\lambda) = f(\lambda, A) = 0$ as $A = a(\lambda)$, at least for $\lambda \in (0,1)$, and

$$a(\lambda) = \frac{-B_1(\lambda) + ((B_1(\lambda))^2 - 4B_0(\lambda)B_2(\lambda))^{1/2}}{2B_2(\lambda)} = \frac{-B_1(\lambda) + \sqrt{D(\lambda)}}{2B_2(\lambda)}.$$

Note that $B_2(\lambda) > 0$ and $B_0(\lambda) < 0$ for $\lambda \in (0,1)$, hence we need to take "+" before the square root to guarantee $a(\lambda) = A > 0$. Along the curve $A = a(\lambda)$, we can calculate $g(\lambda, a(\lambda))$ and get

$$g(\lambda, a(\lambda)) = -\frac{9\lambda(\lambda+1)\sqrt{\lambda}}{20(\lambda+2)^2(\lambda-1)^2(2\lambda+1)^2}[E_1(\lambda) + E_2(\lambda)\sqrt{D_1(\lambda)}],$$

where

$$D_1(\lambda) = 164025D(\lambda)/8\lambda,$$

$$\begin{aligned} E_1(\lambda) = {} & (39200\lambda^{10} + 19600\lambda^9 - 122640\lambda^8 - 70680\lambda^7 + 100170\lambda^6 \\ & + 75261\lambda^5 + 100170\lambda^4 - 70680\lambda^3 - 122640\lambda^2 + 19600\lambda + 39200), \end{aligned}$$

$$E_2(\lambda) = 280\lambda^6 - 420\lambda^5 - 30\lambda^4 + 97\lambda^3 - 30\lambda^2 - 420\lambda + 280.$$

$(E_1(\lambda))^2 - (E_2(\lambda))^2 D_1(\lambda)$ can be factorized as

$$11197440\lambda^5(\lambda-1)^2(\lambda+2)^2(2\lambda+1)^2(10\lambda^2 - 5\lambda - 14)(14\lambda^2 + 5\lambda - 10),$$

which has a unique root at $\bar{\lambda} \approx 0.6852$ for $\lambda \in (0,1)$.

The same hence holds for $g(\lambda, a(\lambda))$ since $E_1(\lambda) > 0$ and $E_2(\lambda)$ has a unique root at $\lambda \approx 0.6278$ for $\lambda \in (0,1)$, with $E_2(\lambda) < 0$ for $\lambda \sim 1$.

Since $F(0) > 0$ and $F(\bar{\lambda}) < 0$, there has to be a zero of F for $\lambda \in (0, \bar{\lambda})$, and no more zeros are possible because $F'(\lambda) < 0$ at all zeros λ of F for $\lambda \in (0, \bar{\lambda})$. Similarly, there can be no zeros of F for $\lambda \in (\bar{\lambda}, 1)$ since $F(\bar{\lambda}) < 0$, $F(\lambda) < 0$ for $0 < 1 - \lambda \ll 1$ and $F'(\lambda) > 0$ at all possible zeros λ of F for $\lambda \in (\bar{\lambda}, 1)$. A numerical calculation shows that the unique solution of $F(\lambda) = 0$ is $\lambda = \lambda^* \approx 0.0549$. $\square$

Since $P_i'(h) \neq 0$ for $i = 1, 2, 3$, there exist functions

$$Q = \tilde{Q}_i(P) = Q(h_i(P)), \tag{11}$$

where $h = h_i(P)$ is the inverse function of $P = P_i(h)$. Hence the interval of definition of (11) is $P \in J_i$, $J_1 = (-1, P_1(0))$, $J_2 = (P_2(0), \lambda)$ and $J_3 = (P_3(0), \frac{\lambda - 1}{3})$.

Let

$$\Sigma_\lambda^{(i)} = \{(P_i, Q_i)(h)\} = \{(P, \tilde{Q}_i(P))\}. \tag{12}$$

By Lemma 3.2, $\tilde{Q}_1'(P) < 0$ for $P \in J_1$ and $\tilde{Q}_2'(P) > 0$ for $P \in J_2$.

Theorem 3.5 (Li [L]). *For all $\lambda \in (0, 1)$ $\tilde{Q}_1''(P) > 0$ for $P \in J_1$, and $\tilde{Q}_2''(P) > 0$ for $P \in J_2$.*

Let us give the outline of the proof of Theorem 3.5. It was proved in [L] that for $i = 1$ and 2 each curve $\Sigma_\lambda^{(i)}$ has the same convexity near its two end points, and $\Sigma_\lambda^{(i)}$ is entirely located inside the triangle formed by the two tangent lines of $\Sigma_\lambda^{(i)}$ at its two end points and the straight line joining these two points. Hence, by an argument in [HI], if the Abelian integral $I(h)$ has more than two zeros for $h > h_i$, then it must have at least four, taking into account the multiplicity, and this contradicts a result in [P].

We remark that by using the same techniques as in [DL1] it is possible to give a different proof of Theorem 3.5.

Theorem 3.6. *For $0 < h \ll 1$, $\tilde{Q}_3''(P) < 0$ for $\lambda \in (0, \lambda^*)$ and $\tilde{Q}_3''(P) > 0$ for $\lambda \in (\lambda^*, 1)$.*

The proof follows from the same method as used in the proof of Lemma 11 (ii) of [DL2], using also Lemma 3.4.

Remark 3.7. For $0 < h \ll 1$ and $\lambda = \lambda^*$ we have $\tilde{Q}_3''(P) < 0$. This will follow from the proof of Lemma 4.6. In fact this point is crucial in the proof of the existence of 5 zeros and of a quadruple zero.

It is obvious that for $h \neq h_i$, $i = 1, 2$, the number of zeros of the Abelian integral (5) is equal to the number of intersection points of the curve $\Sigma_\lambda^{(i)}$ with the straight line

$$\mathscr{L}_{\alpha,\beta} : \alpha + \beta P + Q = 0 \tag{13}$$

in (P, Q)-plane.

By using Lemmas 2.1, 3.1–3.3 and Theorems 3.5 and 3.6 we can already partially represent $\Sigma_\lambda^{(i)}$, $i = 1, 2, 3$ in Fig. 7, where the picture of $\Sigma_\lambda^{(1)}$ and $\Sigma_\lambda^{(2)}$ is complete, and the behaviour of $\Sigma_\lambda^{(3)}$ is so far only known for $\lambda \neq \lambda^*$, $0 < h \ll 1$ and $h \gg 1$. The arrows in the figure correspond to increasing h.

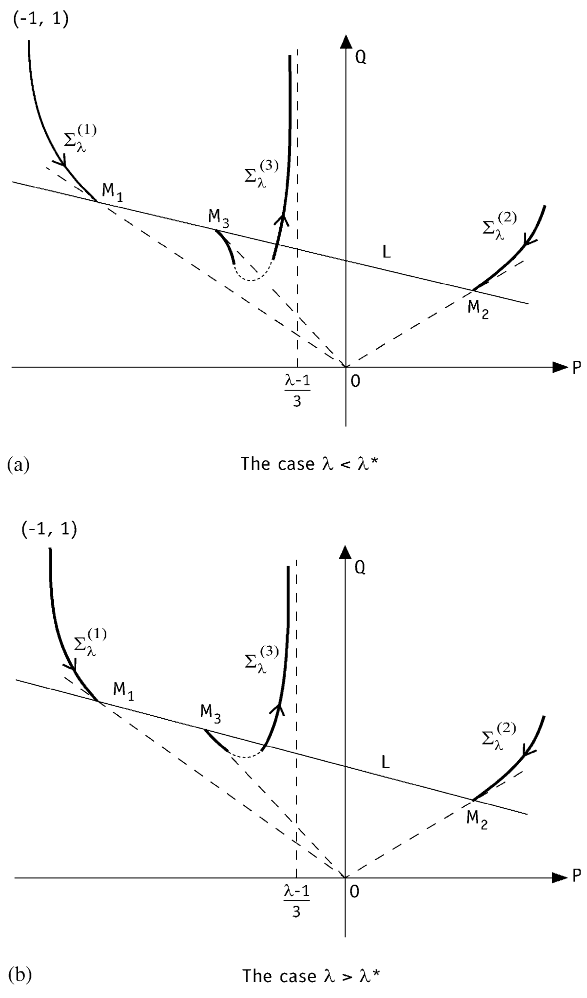

Fig. 7. The behaviour of $\Sigma_\lambda^{(i)}$.

In the rest of the paper we will mainly pay attention to $\Sigma_\lambda^{(3)}$ and prove that for all $\lambda\in(0,1)$, $\Sigma_\lambda^{(3)}$ has a unique minimum point M and there is no inflection point on $\Sigma_\lambda^{(3)}$, right to M; left to M we will find that there is a unique inflection point for $\lambda\in(0,\lambda^*]$, and there are at most two inflection points (or unique quadruple point) for $\lambda\in(\lambda^*,1)$. In particular, there are two inflection points for $0<\lambda-\lambda^*\ll 1$, and there is no inflection point for $0<1-\lambda\ll 1$. These observations will suffice to prove the statements made in Theorem 1.1.

4. The study in (h,ω)-plane

To study the number of zeros of integral (5), we also consider the number of zeros of its second derivative

$$I^{(i)''}(h)=\alpha I_0^{(i)''}(h)+\beta I_1^{(i)''}(h)+I_2^{(i)''}(h). \tag{14}$$

Note that $I_0^{(i)'}(h)$ is the period function of Γ_h^i; $I_0^{(1)''}(h)>0$ and $I_0^{(2)''}(h)>0$ for $h<0$, while $I_0^{(3)''}(h)<0$ for $h>0$, also $\lim_{h\to 0-0} I_0^{(i)''}(h)=+\infty$ for $i=1,2$ and $\lim_{h\to 0+0} I_0^{(3)''}(h)=+\infty$ (see [G]). We define

$$\omega^{(i)}(h)=\frac{I_1^{(i)''}(h)}{I_0^{(i)''}(h)},\quad v^{(i)}(h)=\frac{I_2^{(i)''}(h)}{I_0^{(i)''}(h)}. \tag{15}$$

By taking $a=1$, $b=1-\lambda$ and $c=-\lambda$ in (14) of [DL1], we obtain the differential equation of $(h,\omega^{(i)})$:

$$\begin{cases}\dot h=G(h),\\ \dot\omega=\varphi(h,\omega),\end{cases} \tag{16}$$

where $G(h)$ is the same as in (8), and

$$\begin{aligned}\varphi(h,\omega)=&\left[\frac{3888(\lambda^2+\lambda+1)}{(\lambda-1)(\lambda+2)(2\lambda+1)}h^2-\frac{12(2\lambda^4+\lambda^3-60\lambda^2+\lambda+2)}{\lambda-1}h\right.\\&\left.+\frac{(\lambda+2)(2\lambda+1)(10\lambda^2-11\lambda+10)\lambda^2}{\lambda-1}\right]\omega^2\\&+\left[-\frac{864(5\lambda^2+8\lambda+5)}{(\lambda+2)(2\lambda+1)}h^2+24(\lambda-1)^2(2\lambda^2+7\lambda+2)h\right.\\&\left.+12\lambda^3(\lambda+2)(2\lambda+1)\right]\omega\\&+\frac{144(\lambda-1)(7\lambda^2+13\lambda+7)}{(\lambda+2)(2\lambda+1)}h^2+12(\lambda-1)(\lambda+2)(2\lambda+1)\lambda h.\end{aligned}$$

Similarly, from (21) of [DL1] we obtain the expressions of ω and v by $P(h)$, $Q(h)$ and h:

$$\omega(h) = \frac{(\lambda-1)[108h-(\lambda+2)(2\lambda+1)(12\lambda+14(\lambda-1)P(h)-15Q(h)]}{324h+(\lambda+2)(2\lambda+1)(10\lambda^2-11\lambda+10-12(\lambda-1)P(h))},$$

$$v(h) = \frac{(24(5\lambda^2+8\lambda+5)+360(\lambda-1)P(h)-540Q(h))h+\lambda(\lambda+2)(2\lambda+1)(12\lambda+14(\lambda-1)P(h)-15Q(h))}{324h+(\lambda+2)(2\lambda+1)(10\lambda^2-11\lambda+10-12(\lambda-1)P(h))}. \tag{17}$$

Eq. (16) does not depend on (i). In fact it also holds if we work with Γ_h, recalling that $\Gamma_h = \Gamma_h^3$ for $h \geqslant 0$ and $\Gamma_h = \Gamma_h^1 \cup \Gamma_h^2$ for $h<0$. In the rest of this paragraph we will consider I_0, I_1, I_2, their derivatives and all the related functions $(P, Q, \omega, v, \ldots)$ obtained by integration over Γ_h. We will hence skip the super-indices in (14) and (15).

Let

$$\omega_1 = \frac{(\lambda-1)(2\lambda+7)}{10\lambda^2+31\lambda+31}, \quad \omega_2 = \frac{\lambda(\lambda-1)(7\lambda+2)}{31\lambda^2+31\lambda+10}, \quad \omega_0 = \frac{12\lambda(1-\lambda)}{10\lambda^2-11\lambda+10}, \tag{18}$$

then it is easy to check that for $\lambda \in (0,1)$

$$-1 < \frac{\lambda-1}{3} < \omega_1 < \omega_2 < 0 < \omega_0. \tag{19}$$

From (17) and the Lemmas 2.1 and 3.1 we obtain

Lemma 4.1. (i) $\omega(h_1) = \omega_1$, $\omega(h_2) = \omega_2$, $\omega(0) = 0$;

(ii) $\omega(h) \to \frac{\lambda-1}{3}$ *and* $v(h) \to -\infty$ *as* $h \to +\infty$;

(iii) $\omega(h) < \frac{\lambda-1}{3}$ *for* $h \gg 1$.

System (16) has three saddle points at $S_1(h_1, \omega_1)$, $S_2(h_2, \omega_2)$ and $S_3(0, \omega_0)$, and three nodes at $N_1(h_1, -1)$, $N_2(h_2, \lambda)$ and $O(0,0)$. We denote by C_ω the orbit of (16) satisfying the conditions in Lemma 4.1, then the behaviour of C_ω is shown in Fig. 8. More precisely, we have the following result.

Lemma 4.2. (i) *When* $h_1 < h < 0$, C_ω *is the unstable manifold of the saddle point* S_1, *and the stable manifold of the saddle point* S_2, *it monotonically increases as h increases and tends to the node O tangent to the negative* ω*-axis as* $h \to 0-0$.

(ii) *when* $h>0$, C_ω *tends to the node O tangent to the positive* ω*-axis as* $h \to 0+0$; *it cuts each of the lines* $\{\omega = 0\}$ *and* $\{\omega = \frac{\lambda-1}{3}\}$ *transversally in a unique point, and tends to the line* $\{\omega = \frac{\lambda-1}{3}\}$, *monotonically increasing, for* $h \gg 1$.

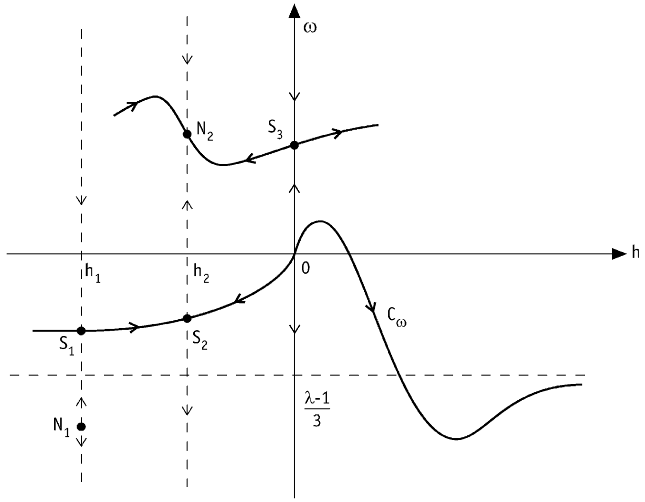

Fig. 8. The behaviour of C_ω.

Proof. The first two parts of conclusion (i) follow by Lemma 4.1(i) and the same analysis as in the proof of Lemma 9 of [DL1]. The last part of (i) and the first part of (ii) follow from the fact that at the point $O(0,0)$ the linearization of system (16) has the matrix

$$12\lambda(\lambda+2)(2\lambda+1)\begin{pmatrix} \lambda^2 & 0 \\ \lambda-1 & \lambda^2 \end{pmatrix}.$$

The last parts of conclusion (ii) follow by Lemma 4.1(ii), (iii) and the fact that for $h>0$ the vector field (16) is strictly pointing downward along the lines $\{\omega=0\}$ and $\left\{\omega=\frac{\lambda-1}{3}\right\}$. It is also possible to prove that C_ω has a unique maximum and a unique minimum for $h>0$, but we do not need it in the further study. $\square$

Taking $a=1$, $b=1-\lambda$ and $c=-\lambda$ in (12) of [DL1] we have

$$v(h)=\frac{12(1-\lambda)h+(36h+\lambda(\lambda+2)(2\lambda+1))\omega(h)}{(1-\lambda)(\lambda+2)(2\lambda+1)}. \tag{20}$$

By definition (15) expression (14) can be changed to the form

$$I''(h)=I_0''(h)(\alpha+\beta\omega(h)+v(h)). \tag{21}$$

Substituting (20) into (21), we have

$$I''(h)=\frac{I_0''(h)}{(1-\lambda)(\lambda+2)(2\lambda+1)}[36(h-\bar{h})\omega(h)-(\lambda-1)(12h+(\lambda+2)(2\lambda+1)\alpha], \tag{22}$$

where

$$\bar{h} = -\frac{1}{36}(\lambda+2)(2\lambda+1)((1-\lambda)\beta+\lambda). \tag{23}$$

If $3\alpha+(\lambda-1)\beta-\lambda=0$, then $12h+(\lambda+2)(2\lambda+1)\alpha = 12(h-\bar{h})$, and (22) becomes

$$I''(h) = \frac{36I_0''(h)}{(1-\lambda)(\lambda+2)(2\lambda+1)}(h-\bar{h})\left(\omega(h)-\frac{\lambda-1}{3}\right),$$

which obviously has two simple zeros $h=\bar{h}$ and $h=\tilde{h}$, where $\tilde{h}$ is the unique value of h such that $\omega(h)=\frac{\lambda-1}{3}$, see Fig. 8.

If

$$3\alpha+(\lambda-1)\beta-\lambda \neq 0, \tag{24}$$

then $h=\bar{h}$ is not a zero of $I''(h)$, and we can, for $h\neq\bar{h}$, rewrite (22) as

$$I''(h) = \frac{36I_0''(h)}{(1-\lambda)(\lambda+2)(2\lambda+1)}(h-\bar{h})(\omega(h)-U(h)), \tag{25}$$

where

$$U(h) = \frac{(\lambda-1)(12h+(\lambda+2)(2\lambda+1)\alpha)}{36(h-\bar{h})}. \tag{26}$$

Note that

$$U'(h) = \frac{(1-\lambda)(\lambda+2)(2\lambda+1)(3\alpha+(\lambda-1)\beta-\lambda)}{108(h-\bar{h})^2}. \tag{27}$$

Under condition (24) the curve $C_U = \{(h,\omega)\,|\,\omega=U(h)\}$ consists of 2 monotone branches, located respectively on the two sides of the line $\{h=\bar{h}\}$ and having the same asymptotic line $\{\omega=\frac{\lambda-1}{3}\}$ as $h\to\pm\infty$. Denote by $C_U^{(1)}$ (respectively, $C_U^{(2)}$) the branch of C_U, located above (respectively, below) the line $\{\omega=\frac{\lambda-1}{3}\}$.

Let $N(C_\omega\cap C_U)$ be the number of intersection points of the curves C_ω and C_U, counting the multiplicity; let S_- be a set in (α,β)-plane such that C_U is monotonically decreasing for $(\alpha,\beta)\in S_-$. By (27)

$$S_-: 3\alpha+(\lambda-1)\beta-\lambda<0. \tag{28}$$

Finally, let $C_\lambda = \{(\alpha,\beta)\,|\,\mathscr{L}_{\alpha,\beta}$ is tangent to $\Sigma_\lambda^{(3)}$ at a point $(P(h),Q(h))\}$.

Since C_ω is a trajectory of system (16), the number $N(C_\omega\cap C_U)$ is controlled by the number of points on C_U, at which the vector field (16) is tangent to C_U. Hence we consider the zeros of $(-U'(h), 1) \cdot (\dot{h}, \dot{\omega})$ along C_U. By (16)

$$\dot{\omega} - U'(h)\dot{h}\big|_{\omega=U(h)} = \frac{(1-\lambda)(\lambda+2)(2\lambda+1)}{(36(h-\bar{h}))^2} A(h), \tag{29}$$

where $A(h) = a_3h^3 + a_2h^2 + a_1h + a_0$, and

$$\begin{cases} a_0 = -\lambda^2(\lambda+2)^2(2\lambda+1)^2[(10\lambda^2 - 11\lambda + 10)\alpha + 12\lambda(1-\lambda)\beta + 12\lambda^2]\alpha, \\ a_1 = 12(\lambda+2)(2\lambda+1)[(2\lambda^4 + \lambda^3 - 60\lambda^2 + \lambda + 2)\alpha^2 \\ +2(\lambda-1)^3(2\lambda^2 + 7\lambda + 2)\alpha\beta - \lambda(\lambda-1)^2(\lambda+2)(2\lambda+1)\beta^2 \\ -2\lambda(\lambda^2 + \lambda + 1)(2\lambda^2 + 11\lambda + 2)\alpha \\ +2\lambda^2(\lambda-1)(2\lambda+1)(\lambda+2)\beta - \lambda^3(\lambda+2)(2\lambda+1)], \\ a_2 = -3888(\lambda^2 + \lambda + 1)\alpha^2 - 864(\lambda-1)(5\lambda^2 + 8\lambda + 5)\alpha\beta \\ -144(\lambda-1)^2(7\lambda^2 + 13\lambda + 7)\beta^2 \\ -288(22\lambda^4 + 29\lambda^3 + 6\lambda^2 + 29\lambda + 22)\alpha \\ -576(\lambda-1)(\lambda^2 - 2\lambda - 2)(2\lambda^2 + 2\lambda - 1)\beta \\ +144\lambda(8\lambda^4 - 11\lambda^3 - 48\lambda^2 - 11\lambda + 8), \\ a_3 = -1728(18\alpha + 6(\lambda-1)\beta + 5\lambda^2 - \lambda + 5). \end{cases} \tag{30}$$

We first consider the region

$$R_+ : \{(\alpha, \beta) \mid a_3 \geqslant 0\}, \tag{31}$$

see Fig. 9. Note that $\{a_3 = 0\}$ is parallel to ∂S_-, hence $R_+ \subset S_-$.

Lemma 4.3. *For any $\lambda \in (0,1)$ and $(\alpha, \beta) \in R_+$ we have $N(C_\omega \cap C_U^{(1)}) \leqslant 2$.*

Proof. By (26) and (23) we have

$$U(0) = \frac{(\lambda-1)\alpha}{(1-\lambda)\beta + \lambda}. \tag{32}$$

We first consider the case $a_3 > 0$.

If $(\alpha, \beta) \in R_+ \cap \{\alpha > 0\}$, then $\beta > 0$ and $U(0) < 0$ by (32). From Lemma 4.2 it follows that C_ω and $C_U^{(1)}$ have exactly 1 intersection point for $h < 0$, counting the multiplicity; and if $N(C_\omega \cap C_U^{(1)}) > 2$, then C_ω and $C_U^{(1)}$ have at least 3 intersection points for $h > 0$, since $C_U^{(1)}$ stays above the line $\{\omega = \frac{\lambda-1}{3}\}$ and C_ω must be below this line for $h \gg 1$. Hence there are at least two "tangent points" on $C_U^{(1)}$ for $h > 0$. Besides, $a_3 > 0$ means that the vector field (16) is pointing upward with respect to $C_U^{(1)}$ for $h \gg 1$, inducing at least one more "tangent point" on $C_U^{(1)}$, right to the most

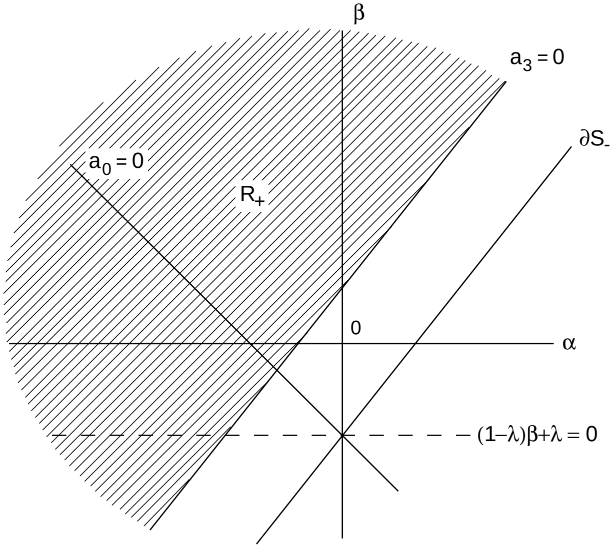

Fig. 9. The region R_+.

right intersection point of C_ω and $C_U^{(1)}$. On the other hand by (4) and (26) in this case we have

$$U(h_2) - \lambda = -\frac{(2\lambda+1)(\alpha+\lambda\beta+\lambda^2)}{(2\lambda+1)\beta+\lambda(3\lambda+1)} < 0, \tag{33}$$

hence there is at least one more tangent point on $C_U^{(1)}$ for $h<0$, contradicting the fact that there exist at most 3 tangent points on C_U (see (29)). The situation is shown in Fig. 10(a).

If $(\alpha,\beta)\in R_+\cap\{\alpha<0\}\cap\{(1-\lambda)\beta+\lambda>0\}$, then by (32) $U(0)>0$, hence $N(C_\omega\cap C_U^{(1)})$ must be even. If this number is bigger than 2, then it is at least 4, implying at least 3 tangent points on $C_U^{(1)}$; by the same argument as above, $a_3>0$ gives at least one more tangent point on $C_U^{(1)}$, also leading to a contradiction, see Fig. 10(b).

If $(\alpha,\beta)\in R_+\cap\{\alpha<0\}\cap\{(1-\lambda)\beta+\lambda\leqslant 0\}$, then by (23) $\bar{h}\geqslant 0$, the discussion being essentially the same as above.

The case $(0,\beta)\in R_+$ is just the critical situation between the cases (a) and (b) in Fig. 10; $C_U^{(1)}$ passes through the origin, and (29) has a root $h=0$, hence the conclusion is obviously true.

Finally, if $a_3=0$, then $A(h)$ has at most two zeros and by the same analysis it is easy to see the validity of the conclusion. □

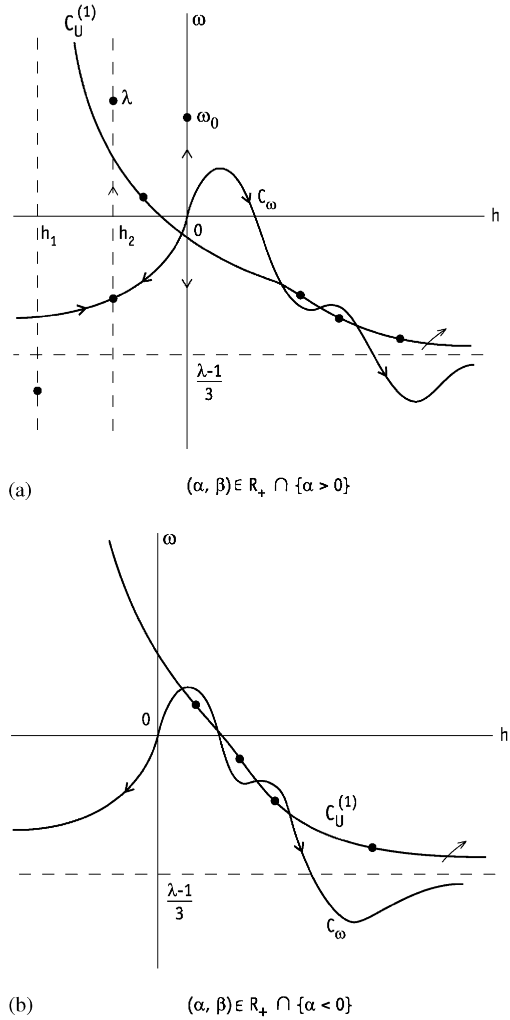

Fig. 10. Hypothetical intersections of C_ω and $C_U^{(1)}$.

Lemma 4.4. *If* $(\alpha, \beta) \in C_\lambda \cap S_-$ *and* $(h, U(h)) \in C_U^{(1)}$, *then for fixed* λ *and* h *we have* $\frac{\partial U}{\partial \alpha} + \frac{\partial U}{\partial \beta} \beta'(\alpha) < 0$.

Proof. Since $(\alpha, \beta) \in C_\lambda$, $\alpha + \beta P(h_0) + Q(h_0) = 0$ for some $h_0 > 0$, hence by Lemma 3.1(ii) we have

$$\beta'(\alpha) = -\frac{1}{P(h_0)} < \frac{3}{1-\lambda}. \tag{34}$$

From (26) we obtain

$$\frac{\partial U}{\partial \alpha}+\frac{\partial U}{\partial \beta}\beta'(\alpha) = \frac{(1-\lambda)(\lambda+2)(2\lambda+1)}{36(h-\bar{h})^2}\left[-36(h-\bar{h})+\frac{(1-\lambda)(12h+(\lambda+2)(2\lambda+1)\alpha)}{-P(h_0)}\right]. \tag{35}$$

If $12h+(\lambda+2)(2\lambda+1)\alpha \leqslant 0$, then the conclusion of Lemma 4.5 is obviously true, since $h-\bar{h}>0$ for $(h,U(h))\in C_U^{(1)}$; if this expression is positive, then by (34) the second factor of the right-hand side of (35) is less than

$$-36(h-\bar{h})+3(12h+(\lambda+2)(2\lambda+1)\alpha)=(\lambda+2)(2\lambda+1)(3\alpha+(\lambda-1)\beta-\lambda)<0,$$

since $(\alpha,\beta)\in S_-$, see (28). $\square$

Let

$$L_0=\{(\alpha,\beta)\mid \alpha+\beta P(0)+Q(0)=0\}. \tag{36}$$

Note that a point $(\alpha,\beta)\in L_0$ corresponds to a straight line in (P,Q)-plane, passing through the point $(P(0),Q(0))$; a point $(\alpha,\beta)\in C_\lambda$ corresponds to a line $\mathscr{L}_{\alpha,\beta}$ that is tangent to $\Sigma_\lambda^{(3)}$ at some point $(P(h),Q(h))\in\Sigma_\lambda^{(3)}$; hence C_λ is a continuous curve in (α,β)-plane, parametrized by h in a way, induced from $\Sigma_\lambda^{(3)}$.

Lemma 4.5.

(i) *C_λ is tangent to L_0 at $h=0$, for any $\lambda\in(0,1)$.*
(ii) *If $\lambda\in(0,\lambda^*)$ (resp. $\lambda\in(\lambda^*,1)$) then for $0<h\ll 1$ C_λ is located above (resp. below) L_0 with positive (resp. negative) curvature (see Figs. 11(a) and (b)).*
(iii) *If $h=\tilde{h}$ corresponds to an inflection point of $\Sigma_\lambda^{(3)}$, then C_λ has a cusp point at $h=\tilde{h}$, and C_λ changes its curvature when h passes through $\tilde{h}$ (see Fig. 11(c)).*

Proof. The curve C_λ is defined by the equations

$$C_\lambda:\begin{cases}\alpha+\beta P+Q=0,\\ \beta\dot{P}+\dot{Q}=0.\end{cases} \tag{37}$$

Since $P'(h)\neq 0$ for all h (Lemma 3.2) we may take P as a parameter on C_λ, changing its defining equations into

$$\begin{cases}\alpha+\beta P+\tilde{Q}(P)=0,\\ \beta+\tilde{Q}'(P)=0,\end{cases} \tag{38}$$

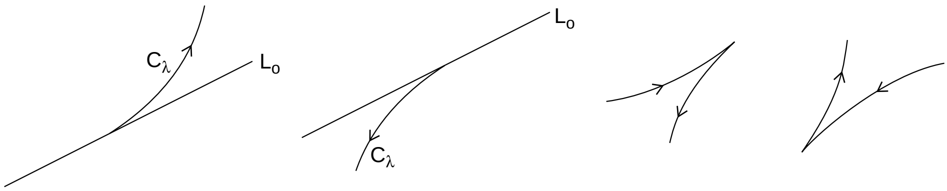

Fig. 11. Behaviour of C_λ.

where $\tilde{Q}(P) = Q(h(P))$ with $h = h(P)$ being the inverse function of $P = P(h)$. Clearly in (α, β)-plane, C_λ is the envelope of the family of straight lines, described by the first equation in (38), where we take P as a parameter. "Fold"-points of C_λ correspond to points where $\tilde{Q}''(P) \neq 0$. In the neighbourhood of such points we can inverse the second equation of (38), writing $P = r(\beta)$ with $\beta = -\tilde{Q}'(r(\beta))$. The equation of C_λ is hence locally given by

$$\alpha = -\beta r(\beta) - \tilde{Q}(r(\beta))$$

and

$$\frac{d\alpha}{d\beta} = -r(\beta).$$

As such

$$\frac{d^2\alpha}{d\beta^2} = \frac{1}{\tilde{Q}''(r(\beta))},$$

and the sign of the curvature of C_λ, seen as a function $\alpha(\beta)$, is given by the sign of $\tilde{Q}''(P)$. Note that $\frac{d^2\beta}{d\alpha^2} = \frac{1}{P^3}\frac{d^2\alpha}{d\beta^2}$. Conclusions (i) and (ii) now follow by Lemmas 3.3 and 3.6, respectively.

At inflection points of $\Sigma_\lambda^{(3)}$ it is easy to see that $\tilde{Q}''(P) = 0$, while $\tilde{Q}'''(P) \neq 0$, and the statement in (iii) easily follows. $\square$

Let

$$h_m = \max\{h > 0 \mid C_\lambda \subset R_+\}. \tag{39}$$

Lemma 4.6.

(i) *If $\lambda \in (0, \lambda^*]$ then $\Sigma_\lambda^{(3)}$ has at most one inflection point for $h \in (0, h_m]$; if the inflection point exists, then no other inflection point nor higher tangency point exists for $h \in (0, +\infty)$.*

(ii) *If* $\lambda > \lambda^*$, *then* $\Sigma_\lambda^{(3)}$ *has at most two inflection points* (*or at most one quadruple point*) *for* $h \in (0, h_m]$; *if we have the occurrence of two inflection points or of one quadruple point for* $h \in (0, h_m]$, *then* $\Sigma_\lambda^{(3)}$ *has no other inflection point nor higher tangency point for* $h \in (0, +\infty)$.

Proof. From the discussion in Section 3 we see that $\Sigma_\lambda^{(3)}$ is parametrized by $h \geqslant 0$ (see (12)). For each $h_T \geqslant 0$, we consider a tangent line $\mathscr{L}_T = \mathscr{L}_{\alpha_T, \beta_T}$ to $\Sigma_\lambda^{(3)}$ at $(P(h_T), Q(h_T))$, i.e. $\alpha_T + \beta_T P(h_T) + Q(h_T) = 0$ and $\beta_T \dot{P}(h_T) + \dot{Q}(h_T) = 0$. Substituting these α_T and β_T into (23) and (26), respectively, we obtain the corresponding $\bar{h}_T$ and $U_T(h)$.

By Lemma 3.3, $(\alpha_T, \beta_T) \to (0, \beta_T^0)$ with $\beta_T^0 > 0$ as $h_T \to 0+0$, hence $(0, \beta_T^0) \in S_-$, see (28) and Fig. 9. Let us study the movement of $C_{U_T}^{(1)}$ in the (h, ω)-plane and the movement of (α_T, β_T) along C_λ in the (α, β)-plane, as h_T increases from 0 to h_m.

If $\lambda \in (0, \lambda^*)$, then by Theorem 3.6, for $0 < h_T \ll 1$, α_T is increasing, hence by Lemma 4.4, $U_T(h)$ is decreasing. This process continues until the first inflection point on $\Sigma_\lambda^{(3)}$ appears, if it exists for some $\tilde{h}_T \in (0, h_m)$. Note that $h = \tilde{h}_T$ is a zero of $I''(h)$ and $\tilde{\alpha}_T > 0$; by (25) and Lemma 4.3 C_ω and $C_{U_T}^{(1)}$ must intersect at the point $(\tilde{h}_T, \omega(\tilde{h}_T))$ (see Fig. 12(b)). As h_T increases from $\tilde{h}_T$ $(0 < h_T - \tilde{h}_T \ll 1)$, α_T decreases (Lemma 4.5), hence by Lemma 4.4 $U_T(h)$ moves up from $\tilde{U}_T(h)$, and the value h of the (unique) intersection point of C_ω and $C_{U_T}^{(1)}$ for $h > 0$ goes down from $\tilde{h}_T$, see Fig. 12(c). This implies that $\Sigma_\lambda^{(3)}$ has no other inflection point for all $h > \tilde{h}_T$, although in the further movement C_ω and $C_{U_T}^{(1)}$ may have 2 intersection points or a tangent point, since at these points the corresponding values of h are less than $\tilde{h}_T$. We can check that Lemma 4.4 remains applicable since we need to stay in region S_- (see (28)) as long as $\tilde{Q}''(P) > 0$. In fact we can check it by proving that $\frac{\partial}{\partial h}(3\alpha + (\lambda - 1)\beta - \lambda) < 0$ for $\tilde{Q}''(P) > 0$, or similarly, since $P'(h) > 0$, that $\frac{\partial}{\partial P}(3\alpha + (\lambda - 1)\beta - \lambda) < 0$. Indeed, by (38) and Lemma 3.1(ii)

$$\begin{aligned} 3\frac{\partial \alpha}{\partial P} + (\lambda - 1)\frac{\partial \beta}{\partial P} &= 3\frac{\partial \alpha}{\partial P} - (\lambda - 1)\tilde{Q}''(P) \\ &= (3P + (1 - \lambda))\tilde{Q}''(P) < 0. \end{aligned}$$

Note that for $h_T > \bar{h}_T$, we only need to pay attention to $C_{U_T}^{(1)}$; the intersection of C_ω and $C_{U_T}^{(2)}$ (which may happen for $(1 - \lambda)\beta + \lambda < 0$) has no influence on our discussion. Note also that for $h \in (0, h_m)$ $C_\lambda \subset R_+ \subset S_-$ and $C_{U_T}^{(1)}$ is a decreasing function of h. Hence as $\lambda \to \lambda^* - 0$, $\tilde{h}_T$ cannot vanish at $h_T = 0$, and the conclusion is still valid for $\lambda = \lambda^*$. This implies the statement made in Remark 3.7.

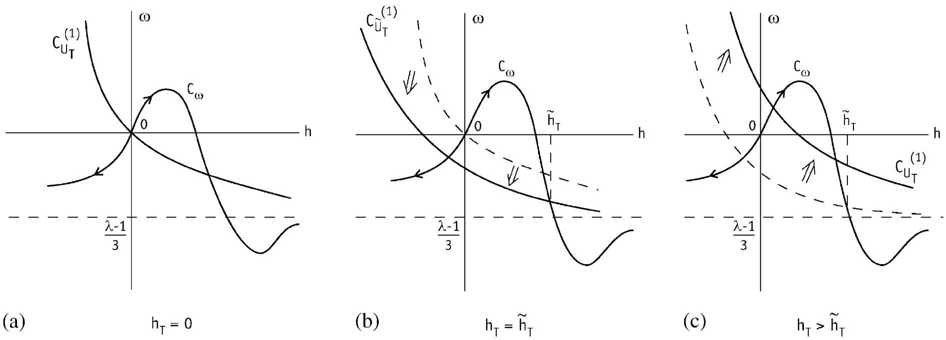

Fig. 12. Movement of $C_{U_T}^{(1)}$ for $\lambda \in (0, \lambda^*)$ as h_T increases.

If $\lambda > \lambda^*$, then by Theorem 3.6, for $0 < h_T \ll 1$, α_T is decreasing as h_T is increasing from 0, hence by the Lemmas 4.4 and 4.5, for any fixed h, $C_{U_T}^{(1)}$ is increasing until the appearance of the first inflection point on $\Sigma_\lambda^{(3)}$, corresponding to $h_T = h_T^{(1)}$. Note that $\alpha_T^{(1)} < 0$, hence by Lemma 4.3 C_ω and $C_{U_T^{(1)}}$ have exactly 2 intersection points for $h > 0$. Since $h - \bar{h}_T > 0$ and $I_0''(h) < 0$ for $h > 0$, by (25), $h_T^{(1)}$ must correspond to the left intersection point, because $I''(h)$ changes its sign from positive to negative as h is increasing and passing through $h_T^{(1)}$. The further movement of $C_{U_T}^{(1)}$ is shown in Fig. 13, where $h_T^{(2)}$ corresponds to the possible second inflection point on $\Sigma_\lambda^{(3)}$ for $h \in (0, h_m]$; it must correspond to the right intersection point at $h_T = h_T^{(2)}$, and there is no possibility to have another inflection point on $\Sigma_\lambda^{(3)}$ for all $h > h_T^{(2)}$, since $C_{U_T}^{(1)}$ moves up from the position of $C_{U_T^{(2)}}^{(1)}$ and never comes back. We remark here that the second inflection point may appear for $h > h_m$, see the proof of Lemma 5.6.

Finally, it is possible, as h_T increases from 0, that $C_{U_T}^{(1)}$ is increasing from the position of Fig. 13(a); then $C_{U_T}^{(1)}$ is tangent to C_ω at some point for $h = \hat{h}_T$ which corresponds to a quadruple point on $\Sigma_\lambda^{(3)}$, and $C_{U_T}^{(1)}$ is continuously increasing as h_T increases from $\hat{h}_T$; hence the quadruple point must be unique for all $h \in (0, +\infty)$. $\square$

5. The study in (h, P, Q)-space and in (h, P)-plane

Since it is impossible to extend the conclusion of Lemma 4.3 from $(\alpha, \beta) \in R_+$ to the region $\mathbb{R}^2 \backslash R_+$, we will use another technique to study the number of zeros of $I(h)$ for $(\alpha, \beta) \in \mathbb{R}^2 \backslash R_+$.

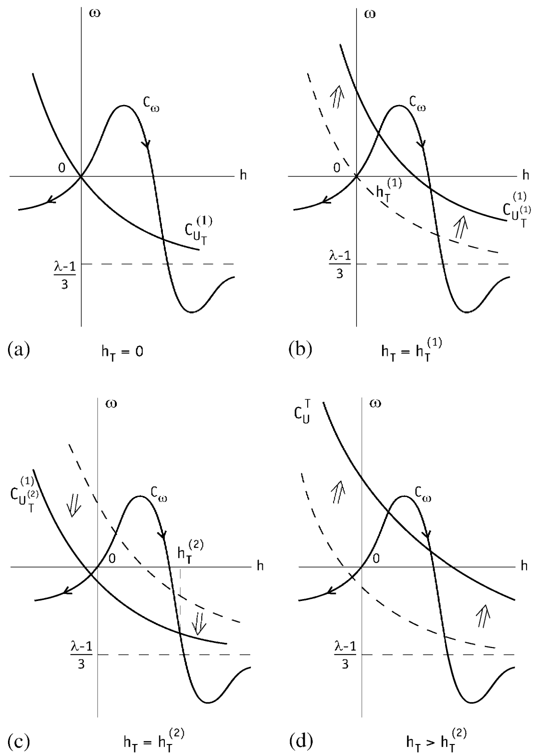

Fig. 13. Movement of $C_{U_T}^{(1)}$ for some $\lambda > \lambda^*$ as h_T increases.

Suppose that integral (5) has k zeros for $h > 0$, counting the multiplicity, i.e. $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) = k$, with $k \geqslant 2$. This is equivalent to the fact that in (h, P, Q)-space the trajectory of system (8) $\tilde{\Sigma}_\lambda = \{(h, P, Q) \mid P = P(h),\ Q = Q(h),\ h > 0\}$ and the plane $\tilde{\mathscr{L}}_{\alpha,\beta} = \{(h, P, Q) \mid (P, Q) \in \mathscr{L}_{\alpha,\beta}\}$ have k intersection points, counting the multiplicity; we will restrict to $h > 0$. Hence there are at least $k - 1$ points $\tilde{M}_i \in \tilde{C}_P = \{(h, P, Q) \in \tilde{\mathscr{L}}_{\alpha,\beta} \mid P = P(h)\}$, at which the vector field (8) is tangent to $\tilde{\mathscr{L}}_{\alpha,\beta}$ (cf. [DL3, Fig. 8]). $\{\tilde{M}_i\}$ are given by the equations

$$\begin{cases} \alpha + \beta P + Q = 0, \\ \beta \dot{P} + \dot{Q} = 0. \end{cases} \tag{40}$$

By using (8) and eliminating Q from (40) we obtain

$$X(h) - Y(h)P = 0, \tag{41}$$

where $X(h) = x_2h^2 + x_1h + x_0,\ Y(h) = y_2h^2 + y_1h + y_0$, and

$$\begin{cases} x_2 = 144(6\alpha + (\lambda-1)\beta + \lambda^2 + \lambda + 1), \\ x_1 = -180(\lambda^2+\lambda+1)\alpha^2 - 180(\lambda-1)(\lambda+1)^2\alpha\beta \\ -24(5\lambda^4 + 3\lambda^3 - 7\lambda^2 + 3\lambda + 5)\alpha \\ -12\lambda(\lambda-1)(\lambda+2)(2\lambda+1)\beta + 12\lambda^2(\lambda+2)(2\lambda+1), \\ x_0 = -3\lambda^2(\lambda+2)(2\lambda+1)(5\alpha+4\lambda)\alpha, \\ y_2 = -144(3\beta + 2(\lambda-1)), \\ y_1 = 180(\lambda^2+\lambda+1)\alpha\beta + 180(\lambda-1)(\lambda+1)^2\beta^2 + 24(\lambda-1)(\lambda^2-5\lambda+1)\alpha \\ +12(14\lambda^4 - 3\lambda^3 - 31\lambda^2 - 3\lambda + 14)\beta - 24\lambda(\lambda-1)(7\lambda^2+13\lambda+7), \\ y_0 = \lambda^2(\lambda+2)(2\lambda+1)(14(\lambda-1)+15\beta)\alpha. \end{cases} \tag{42}$$

In the sequel we will suppose that $X(h)$ and $Y(h)$ have no common factor. If $X(h)$ and $Y(h)$ have two common factors, i.e. $\frac{x_2}{y_2} = \frac{x_1}{y_1} = \frac{x_0}{y_0}$, then calculations show that (α, β) must take 1 of 7 values, for which $Y(h) = 0$ has at least 1 negative or 0 solution, hence (41) has at most 2 solutions for $h > 0$, since $P'(h) > 0$. This is not interesting in our further study (in the proof of Lemma 5.8 and Theorem 5.9, we will only need to consider the case that (41) has more than 2 solutions for $h > 0$). If for some $h_0 > 0,\ Y(h_0) = 0$ and $X(h_0) \neq 0$, then h_0 is not a solution of (41), because $-1 < P(h) < \frac{\lambda-1}{3} < 0$ is bounded. If h_0 is a simple zero for both $X(h)$ and $Y(h)$, then (41) takes a form $P = R(h)$, where $R(h)$ is a ratio of two linear functions of h, the discussion will then be similar to the one below, but much simpler. So we suppose that $X(h)$ and $Y(h)$ have no common factor and the roots of $Y(h)$ (if existing for $h > 0$) are not a solution of (41). Thus (41) becomes

$$P = R(h) = \frac{X(h)}{Y(h)}. \tag{43}$$

From the discussion above, $\tilde{M}_i \in \tilde{C}_P \cap \tilde{C}_R$, where $\tilde{C}_R = \{(h, P, Q) \in \mathscr{L}_{\alpha,\beta} \mid P = R(h)\}$. The number $N(\tilde{C}_P \cap \tilde{C}_R)$ is at least one more than the number of points on $\tilde{C}_R$, at which the vector field (8) is tangent to $\tilde{C}_R$.

Hence we consider

$$\dot{P} - R'(h)\dot{h}\Big|_{\substack{P=R(h) \\ Q=-\alpha-\beta R(h)}} = \frac{G(h)B(h)}{Y^2(h)}, \tag{44}$$

where $G(h) = \dot{h} > 0$ for $h > 0$, $B(h) = b_3h^3 + b_2h^2 + b_1h + b_0$, and

$$\begin{cases} b_3 = (3\beta + 2(\lambda - 1))a_3, \\ b_2 = -58320(\lambda^2 + \lambda + 1)\alpha^2\beta - 12960(\lambda - 1)(5\lambda^2 + 8\lambda + 5)\alpha\beta^2 \\ -2160(\lambda - 1)^2(7\lambda^2 + 13\lambda + 7)\beta^3 + 12960(\lambda - 1)(5\lambda^2 + 17\lambda + 5)\alpha^2 \\ -864(14\lambda^4 - 47\lambda^3 + 120\lambda^2 - 47\lambda + 14)\alpha\beta \\ -288(\lambda - 1)(109\lambda^4 + 203\lambda^3 + 204\lambda^2 + 203\lambda + 109)\beta^2 \\ +576(\lambda - 1)(86\lambda^4 + 322\lambda^3 + 34\lambda^2 + 322\lambda + 86)\alpha \\ -144(112\lambda^6 + 72\lambda^5 - 27\lambda^4 - 152\lambda^3 - 27\lambda^2 + 72\lambda + 112)\beta \\ +288\lambda(\lambda - 1)(56\lambda^4 + 187\lambda^3 + 270\lambda^2 + 187\lambda + 56), \\ b_1 = 12(\lambda + 2)(2\lambda + 1)[-180(\lambda - 1)(\lambda^2 + \lambda + 1)\alpha^3 \\ -15(34\lambda^4 - \lambda^3 - 12\lambda^2 - \lambda + 34)\alpha^2\beta \\ -30\lambda(\lambda - 1)(10\lambda^4 + 3\lambda^3 - 8\lambda^2 + 3\lambda + 10)\alpha\beta^2 \\ +15\lambda(\lambda - 1)^2(10\lambda^2 + 19\lambda + 10)\beta^3 - 10(\lambda - 1)^3(14\lambda^2 + 5\lambda + 14)\alpha^2 \\ -2(140\lambda^6 - 160\lambda^5 + 55\lambda^4 - 43\lambda^3 + 55\lambda^2 - 160\lambda + 140)\alpha\beta \\ +2\lambda(\lambda - 1)(70\lambda^4 - 85\lambda^3 - 249\lambda^2 - 85\lambda + 70)\beta^2 + 4\lambda(\lambda - 1) \\ (70\lambda^4 + 65\lambda^3 + 126\lambda^2 + 65\lambda + 70)\alpha \\ -\lambda^2(280\lambda^4 - 58\lambda^3 - 417\lambda^2 - 58\lambda + 280)\beta \\ +2\lambda^3(\lambda - 1)(70\lambda^2 + 121\lambda + 70)], \\ b_0 = 5\lambda^2(\lambda + 2)^2(2\lambda + 1)^2\bar{b}_0\alpha^2, \end{cases} \tag{45}$$

with $\bar{b}_0 = 36(1 - \lambda)\alpha - 3(10\lambda^2 - 11\lambda + 10)\beta + 2(1 - \lambda)(14\lambda^2 - \lambda + 14)$, and a_3 is the same as in (30).

The projection $(h, P, Q) \mapsto (h, P)$, from the plane $\tilde{\mathscr{L}}_{\alpha,\beta}$ onto the (h, P)-plane, is obviously one to one, hence $N(\tilde{C}_P \cap \tilde{C}_R) = N(C_P \cap C_R)$, where $C_P = \{(h, P) \mid P = P(h)\}$ and $C_R = \{(h, P) \mid P = R(h)\}$ are images of $\tilde{C}_P$ and $\tilde{C}_R$, respectively. For simplicity we will say "the tangent point on C_R" (with respect to the vector field (8)), which in fact makes sense only in (h, P, Q)-space as we explained above.

From (42) we see that $y_2 = 0$ is only occurring on the straight line $\{\beta = \frac{2}{3}(1 - \lambda)\}$ which we will treat separately in Lemma 5.7. If $y_2 \neq 0$, then $R(h) \to \frac{x_2}{y_2}$ as $h \to \pm\infty$, and

$$\tau = \frac{x_2}{y_2} - \frac{\lambda - 1}{3} = -\frac{a_3}{36y_2} = \frac{a_3}{5184(3\beta - 2(1 - \lambda))}. \tag{46}$$

C_R consists of 1, 2 or 3 branches, depending on whether $Y(h) = 0$ has respectively zero, one or two real roots.

By (43) and (42) the following Lemma is obviously true.

Lemma 5.1. *For any constants a, b and c, the straight line $\{P = ah + b\}$ (resp. $\{P = c\}$) cuts the curve C_R at most at 3 (resp. 2) points, counting the multiplicity.*

Fig. 14 shows the behaviour of C_R, where the horizontal straight line is the asymptotic line $\{P = \frac{x_2}{y_2}\}$ and the vertical straight line(s) correspond to the zero(s) of $Y(h) = 0$.

Note that only in cases (a_i) the curve C_R has more than one inflection point (we do not represent the "symmetric" cases for (a_i) in Fig. 14); in other cases, the unique inflection point appears on one of the branches of (b_i), on one of the side branches of (c1) and (c2) and on the middle branch of (c3) and (c4).

Let $L_a = \{(\alpha, \beta) \mid a_3 = 0\}$ and $L_b = \{(\alpha, \beta) \mid \bar{b}_0 = 0\}$. Denote by k_a, k_b, k_0 the slope of L_a, L_b and L_0, respectively, and by α_a, α_b, α_0 the α-coordinate of the intersection point of L_a, L_b and L_0 with $\{\beta = 0\}$ respectively; we recall that L_0 has been defined in (36).

Lemma 5.2. *For any $\lambda \in (0, 1)$, $0 < k_b < k_0 < k_a$, $\alpha_b < \alpha_0 < \alpha_a$; $k_0 > 1$ and $\alpha_0 > -1$.*

Proof. By (45), (36) and (30) we have

$$\begin{cases} k_b = \dfrac{12(1-\lambda)}{10\lambda^2 - 11\lambda + 10}, & k_0 = -\dfrac{1}{P(0)} = -\dfrac{I_0^{(3)}(0)}{I_1^{(3)}(0)}, \quad k_a = \dfrac{3}{1-\lambda}, \\[2ex] \alpha_b = -\dfrac{14\lambda^2 - \lambda + 14}{18}, & \alpha_0 = -Q(0) = -\dfrac{I_2^{(3)}(0)}{I_0^{(3)}(0)}, \quad \alpha_a = -\dfrac{5\lambda^2 - \lambda + 5}{18}. \end{cases} \tag{47}$$

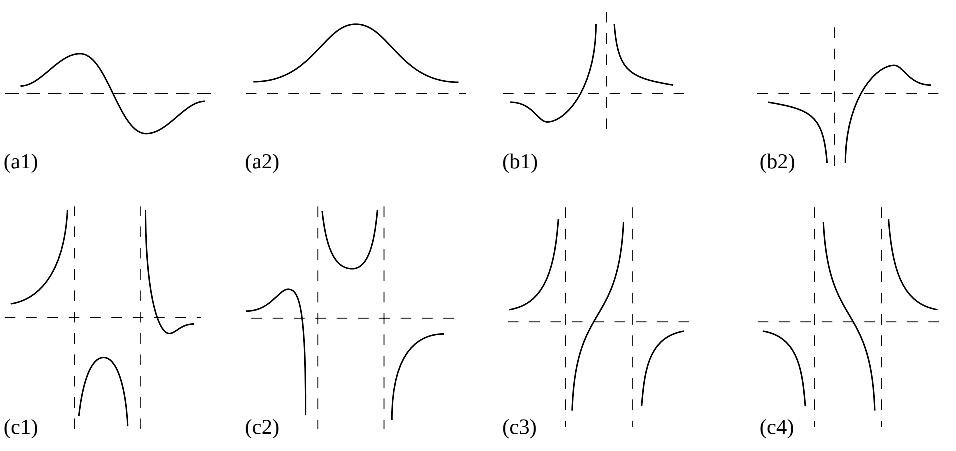

Fig. 14. The behaviour of the curve C_R.

Since $-1 < P(0) < \frac{\lambda-1}{3} < 0$ (see Lemma 3.1 and Fig. 7) we immediately obtain

$$k_a - k_0 = \frac{\lambda - 1 - 3P(0)}{(\lambda-1)P(0)} > 0, \quad k_0 - 1 = \frac{P(0)+1}{-P(0)} > 0.$$

By using Lemma 2.2 and (47) a calculation shows

$$k_0 - k_b = -\frac{12\lambda^2\sqrt{\lambda}}{(10\lambda^2 - 11\lambda + 10)I_1^{(3)}(0)} > 0, \quad \alpha_0 - \alpha_b = \frac{14\lambda^2\sqrt{\lambda}}{15I_0^{(3)}(0)} > 0.$$

Note that $I_0^{(3)}(0) > 0$, $I_1^{(3)}(0) < 0$ and $I_2^{(3)}(0) > 0$. By a similar calculation we have

$$\begin{aligned}
&\alpha_a - \alpha_0 \\
&= \frac{6\sqrt{\lambda}(5\lambda^4 + 5\lambda^3 - 11\lambda^2 + 5\lambda + 5) + 5\sqrt{2}(1-\lambda)(\lambda+2)(2\lambda+1)(\lambda^2+1)\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}}}{135I_0^{(3)}(0)} \\
&> 0
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_0 + 1 \\
&= \frac{6\sqrt{\lambda}(-70\lambda^4 - 65\lambda^3 + 144\lambda^2 + 25\lambda + 20) + 5\sqrt{2}(1-\lambda)(\lambda+2)(7\lambda-4)(2\lambda+1)^2\arctan\frac{\sqrt{2}(1-\lambda)}{3\sqrt{\lambda}}}{1215I_0^{(3)}(0)} \\
&> 0.
\end{aligned}$$

Note that only in the last expression we need some extra calculations like in the proof of Lemma 3.4; the minimum value of $\alpha_0 + 1$ for $\lambda \in [0, 1]$ is achieved at $\lambda = 1$; it is $\frac{1}{5}$. $\square$

The straight lines L_a and L_b cross at the point

$$(\alpha_1, \beta_1) = \left(\frac{2\lambda^4 - 17\lambda^3 + 3\lambda^2 - 17\lambda + 2}{18(2\lambda^2 - \lambda + 2)}, \frac{2(1-\lambda)(\lambda^2+1)}{2\lambda^2 - \lambda + 2}\right). \tag{48}$$

For $\lambda \in (0, 1)$, $\alpha_1 = 0$ has a unique root at $\lambda_1 \approx 0.1185$. A calculation shows that $\beta_0 - \beta_b > 0$ and $\beta_a - \beta_0 = 0$ has a unique root for $\lambda \in (0, 1)$ at $\lambda_2 \approx 0.1204$; β_a, β_b and β_0 are, respectively, the β-coordinate of the intersection point of L_a, L_b and L_0 with $\{\alpha = 0\}$. By using this fact as well as Lemma 5.2 we immediately obtain the following result.

Lemma 5.3. *For different $\lambda \in (0, 1)$ there are three different relative positions of L_a, L_b and L_0, shown in Fig. 15; in any case $L_0 \subset R_+ \cup G$, where G is the region in (α, β)-plane, formed by $a_3 < 0$ and $\bar{b}_0 \leqslant 0$.*

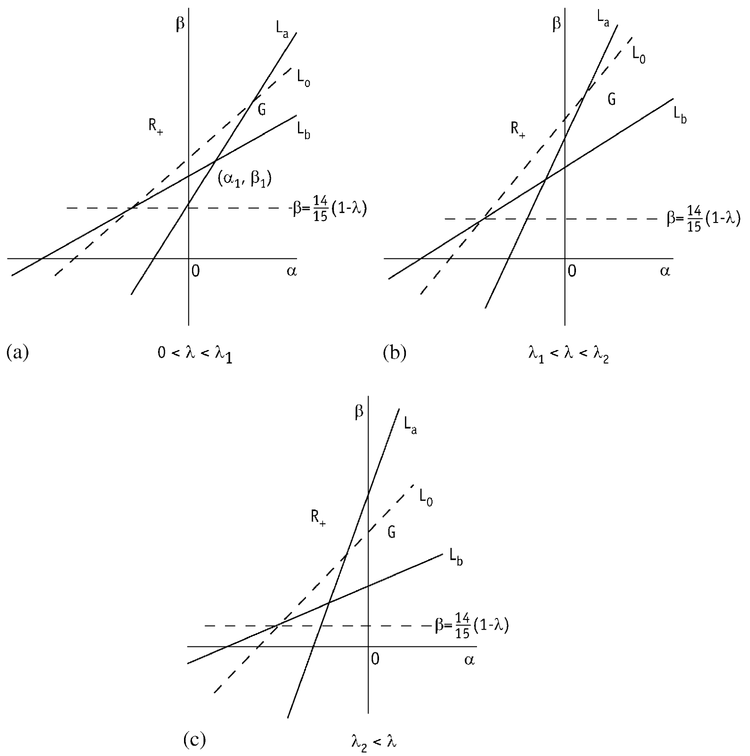

Fig. 15. Relative positions of L_a, L_b and L_0.

Lemma 5.4. (i) *If $\mathscr{L}_{\alpha,\beta}$ and $\Sigma_\lambda^{(3)}$ have a tangency of order k $(\geqslant 2)$ at $h = \tilde{h}$, then C_P and C_R have a tangency of order $k-1$ at the same $\tilde{h}$ (tangency of order 1 means transversal) intersection.*

(ii) *If $\Sigma_\lambda^{(3)}$ has an inflection point or a higher tangency point at $h = \tilde{h}$, with tangent line $\mathscr{L}_{\alpha,\beta}$, then $B(\tilde{h}) = 0$.*

(iii) *If $\mathscr{L}_{\alpha,\beta}$ and $\Sigma_\lambda^{(3)}$ have $\ell(\geqslant 3)$ intersection points, then $B(h)$ has at least $\ell - 2$ positive zeros.*

Proof. At $(P(\tilde{h}), Q(\tilde{h}))$, $\tilde{\mathscr{L}}_{\alpha,\beta}$ and $\tilde{\Sigma}_\lambda^{(3)}$ have a tangency of order k, hence at the same point $\tilde{C}_P$ and $\tilde{C}_R$ have a tangency of order $k-1$, this proves (i). The conclusion (ii) is just a consequence of (i) and (44). The conclusion (iii) is straightforward by the construction of C_R. $\square$

Lemma 5.5. *For any $\lambda \in (0,1)$*

$$P(0) - R(0) = \frac{15(\alpha + \beta P(0) + Q(0))}{15\beta - 14(1-\lambda)}. \tag{49}$$

Proof. From (43) and (42) we have $R(0) = \frac{-3(5\alpha+4\lambda)}{15\beta-14(1-\lambda)}$. By using (9) it is easy to get (49). □

Lemma 5.6. *On its part where* $C_\lambda \subset G$, $\Sigma_\lambda^{(3)}$ *has at most one inflection point for* $\lambda \in (0, \lambda^*]$ *and at most two inflection points* (*or at most one quadruple point*) *for* $\lambda \in (\lambda^*, 1)$; *if this maximal degeneracy occurs then there are no other points on* $\Sigma_\lambda^{(3)}$ *with tangency higher than two.*

Proof. We first consider the case $\lambda \in (0, \lambda^*]$. The position of the region G is shown in Fig. 15(a), where $M_1(\alpha_1, \beta_1)$ is given by (48). Let us from now on denote by C_{a_i} or C_{b_i}, respectively, the curve in (α, β)-plane given by $\{a_i = 0\}$ or $\{b_i = 0\}$. Using (30) and (45) it is not difficult to find (by Maple, for example) that for $\alpha \geqslant \alpha_1$ the curves C_{a_2} and C_{b_2} intersect L_a only once and at the same point $M_2(\alpha_2, \beta_2)$, with

$$\begin{cases} \alpha_2 = \dfrac{13\lambda^6 - 51\lambda^5 - 318\lambda^4 - 503\lambda^3 - 318\lambda^2 - 51\lambda + 13}{108(\lambda+2)(2\lambda+1)(\lambda^2+\lambda+1)}, \\ \beta_2 = \dfrac{73\lambda^6 + 147\lambda^5 - 30\lambda^4 - 137\lambda^3 - 30\lambda^2 + 147\lambda^2 + 73}{36(1-\lambda)(\lambda+2)(2\lambda+1)(\lambda^2+\lambda+1)}. \end{cases} \tag{50}$$

Note that $\{C_{a_i}\}$ are conic curves, and $a_1 < 0$ for all $(\alpha, \beta) \in G(\alpha > 0)$ and $\lambda \in (0, \lambda_3)$, where $\lambda_3 \approx 0.36359$ (the case $\lambda \geqslant \lambda_3$ will be discussed later on). Let $G_1 \subset G$ be the curved triangle: $\{(\alpha, \beta) \mid a_3 < 0, \bar{b}_0 \leqslant 0, a_2 \geqslant 0\}$. Then $a_i < 0$ for all $i = 0, 1, 2, 3$ if $(\alpha, \beta) \in G \backslash G_1$, implying that $A(h) = 0$ has no positive solution in this region, so that the conclusions of the Lemmas 4.3 and 4.6 are valid. So we will restrict to $(\alpha, \beta) \in G_1$. C_{b_1} cuts L_a and C_{a_2}, respectively, at one point, while $C_{b_1} \cap L_b = \emptyset$ for $(\alpha, \beta) \in G_1$. $B(h)$ has two positive zeros and one negative zero for $(\alpha, \beta) \in G_1$ and near (α_2, β_2), it has only one negative zero for $(\alpha, \beta) \in C_{b_1}$ and has three negative zeros for $(\alpha, \beta) \in L_b$ (the last fact is easy to see, because $b_0 < 0$, $b_1 = 0$, $b_2 < 0$, $b_3 < 0$ on C_{b_1}, and all $\{b_i\}$ are negative on L_b). As such the double zero curves of $B(h) = 0$ must lay in G_1 above and below C_{b_1}, denoted, respectively, by C_Δ^1 and C_Δ^2. More precisely, the double zero bifurcation curve of $B(h) = 0$ is given by

$$\Delta = 27b_3^2b_0^2 - b_2^2b_1^2 + 4b_3b_1^3 + 4b_2^3b_0 - 18b_3b_2b_1b_0 = 0. \tag{51}$$

Substituting (45) into (51) we obtain $\Delta = \Delta_1(\alpha, \beta) \cdot \Delta_2(\alpha, \beta)$, with

$$\begin{aligned} \Delta_1(\alpha, \beta) = {} & 8100(\lambda^2+\lambda+1)\alpha^2\beta + 675(\lambda-1)(14\lambda^2+23\lambda+14)\alpha\beta^2 \\ & + 225(\lambda-1)^2(10\lambda^2+19\lambda+10)\beta^3 - 3240(\lambda-1)(\lambda^2+5\lambda+1)\alpha^2 \end{aligned}$$

$$+\,180(34\lambda^4-19\lambda^3+24\lambda^2-19\lambda+34)\alpha\beta+15(\lambda-1)(280\lambda^4$$

$$326\lambda^3+219\lambda^2+326\lambda+280)\beta^2-180(\lambda-1)(14\lambda^4+83\lambda^3+94\lambda^2$$

$$+\,83\lambda+14)\alpha+4(2\lambda^2-\lambda+2)(245\lambda^4-35\lambda^3-339\lambda^2-35\lambda+245)\beta$$

$$-\,4\lambda(\lambda-1)(7\lambda^2+4\lambda+7)(70\lambda^2+121\lambda+70).$$

$\Delta_1(\alpha,\beta)$ corresponds to C_Δ^1, while $\Delta_2(\alpha,\beta)$ corresponds to C_Δ^2 and another branch C_Δ^3, which is tangent to L_a at (α_2,β_2) and stays outside G, as represented in Fig. 16(a). We do not give the expression of $\Delta_2(\alpha,\beta)$, because we will not use C_Δ^2 and C_Δ^3. From the expression of $\Delta_1(\alpha,\beta)$ it is not difficult to find that along C_Δ^1 the β-coordinate is monotonically increasing with respect to α.

Let $\tilde{G}_1=\{(\alpha,\beta)\in G_1 \mid (\alpha,\beta)$ is on and above $C_\Delta^1\}$. Then $B(h)$ has no positive zero for $(\alpha,\beta)\in G_1\backslash\tilde{G}_1$.

Suppose that the first inflection point of $\Sigma_\lambda^{(3)}$ corresponds to a point $M(\alpha,\beta)\in C_\lambda\subset\tilde{G}_1$ at $h=\tilde{h}$. By Lemma 4.5, M must be above L_0, hence $\alpha+\beta P(0)+Q(0)<0$ (since $Q(0)<0$). By (48) $\beta_1-\frac{14}{15}(1-\lambda)=\frac{2(1-\lambda)(\lambda^2+7\lambda+1)}{15(2\lambda^2-\lambda+2)}>0$ for $\lambda\in(0,1)$. Thus $P(0)-R(0)<0$ by (49). Suppose that $\mathscr{L}_{\alpha,\beta}$ is tangent to $\Sigma_\lambda^{(3)}$ at $h=h_T$ and let us study the relative positions of C_P and C_R for h_T increasing and passing through $\tilde{h}$. By Lemma 5.4 C_P must intersect C_R at $h=h_T$ and C_P is tangent to C_R at $h=\tilde{h}$. Besides, we have $B(\tilde{h})=0$. Since $P(0)<R(0)$ and $b_0<0$, one more zero of $B(h)$ must exist at a positive value $\hat{h}<\tilde{h}$, see Fig. 17(a).

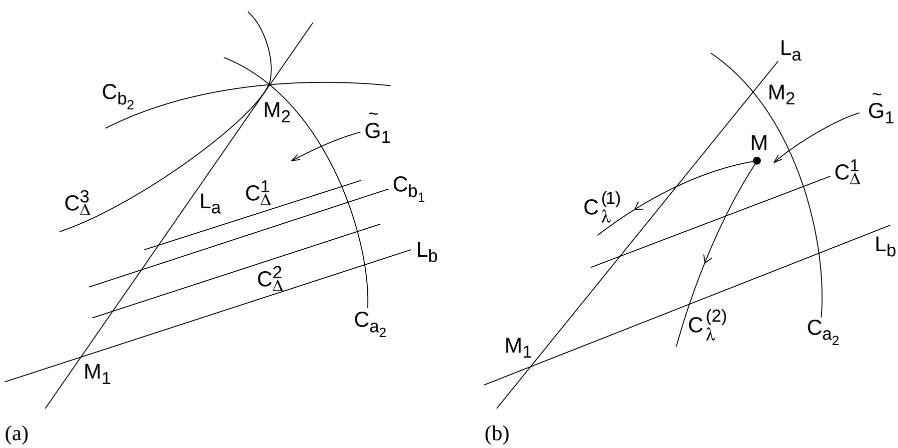

Fig. 16. Region $\tilde{G}_1$.

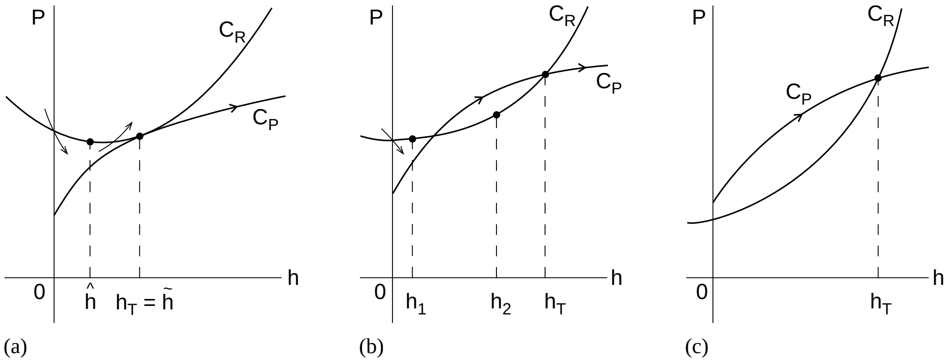

Fig. 17. The relative positions of C_P and C_R for $\lambda \in (0, \lambda^*]$.

When h_T increases from $\tilde{h}$, the two positive zeros of $B(h)$ exist and both of them are less than h_T, as long as (α_T, β_T) is still above L_0, see Fig. 17(b). By Lemma 5.4(ii), $\Sigma_\lambda^{(3)}$ has no other inflection point nor higher tangency point, since in region G, $B(h)$ has at most two positive zeros ($b_3 < 0, b_0 \leqslant 0$). Thus, by Lemma 4.5 (α_T, β_T) moves along C_λ in the same direction (both α and β are decreasing) and C_λ keeps the same convexity. After (α_T, β_T) crosses L_0, there are two possibilities: either C_λ crosses L_a into the region R_+ (see Fig. 16(b)), then by the discussion in the proof of Lemma 4.6, the conclusion of this lemma follows immediately; or C_λ crosses C_Δ^1 into $G_1 \backslash \tilde{G}_1$, then by the discussion above, $B(h)$ has no positive zero, and the conclusion follows by Lemma 5.4(ii).

The proof for $\lambda \in (\lambda^*, 1)$ is essentially the same. We only mention the following differences.

(1) When $\lambda > \lambda_3$ the curve C_{a_1} moves into G from the left side. For $(\alpha, \beta) \in G$ and left to C_{a_1}, we have $a_3 < 0$, $a_0 > 0$ and $a_1 > 0$ (note that $\alpha < 0$), this implies that $A(h)$ has only one positive zero, and the conclusion of Lemma 4.3 (hence the conclusion of Lemma 4.6) is valid.

(2) By Lemma 4.5, corresponding to the first inflection point of $\Sigma_\lambda^{(3)}$ at $\tilde{h}_1$, the point $\tilde{M}_1(\tilde{\alpha}_1, \tilde{\beta}_1) \in C_\lambda \subset \tilde{G}_1$ is below L_0, hence $P(0) > R(0)$, and the other positive zero of $B(h)$ is bigger than $\tilde{h}_1$, see Fig. 18(a). By Lemma 4.5, as $h_T > \tilde{h}_1$, (α_T, β_T) moves up along C_λ. If (α_T, β_T) moves into R_+, then the conclusion follows from the discussion in the proof of Lemma 4.6; if (α_T, β_T) stays in $\tilde{G}_1$, then it is necessary to have a value $\tilde{h}_2$ which corresponds to the second inflection point of $\Sigma_\lambda^{(3)}$, see Fig. 18(b), and when $h_T > \tilde{h}_2$, the situation is exactly the same as the one discussed above, see Fig. 18(c).

(3) As λ increases the curves C_Δ^1 and C_{b_1} in Fig. 16(a) move up, and when $\lambda > 0.5$, C_{b_1} moves out of the region G_1, hence the corresponding $\Sigma_\lambda^{(3)}$ has no inflection point nor higher tangency point for $(\alpha, \beta) \in G$. □

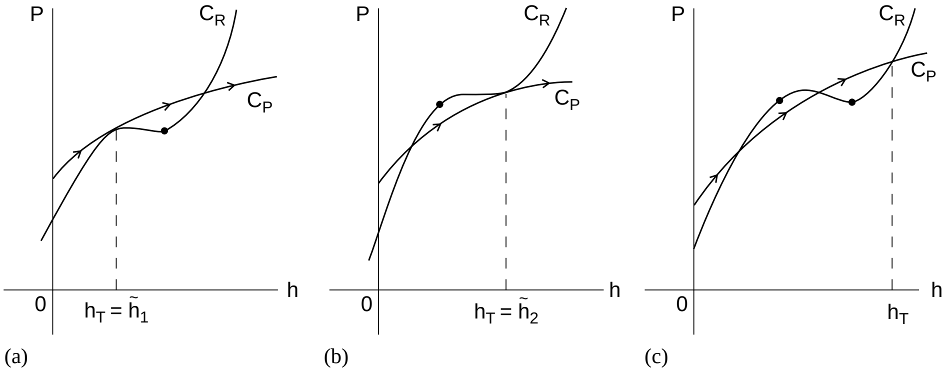

Fig. 18. The relative positions of C_P and C_R for $\lambda > \lambda^*$.

Let $\tilde{L} = \{(\alpha, \beta) \mid a_3 < 0 \text{ and } \beta = \frac{2}{3}(1-\lambda)\}$, $F_1 = \{(\alpha, \beta) \mid a_3 < 0, \bar{b}_0 > 0, \beta > \frac{2}{3}(1-\lambda)\}$ and $F_2 = \{(\alpha, \beta) \mid a_3 < 0, \beta < \frac{2}{3}(1-\lambda)\}$, then $\mathbb{R}^2 \backslash (R_+ \cup G) = F_1 \cup \tilde{L} \cup F_2$.

Lemma 5.7. *For any $\lambda \in (0,1)$, $B(h)$ has at most one positive zero for $(\alpha, \beta) \in F_1 \cup \tilde{L}$ and at most two positive zeros for $(\alpha, \beta) \in F_2$.*

Proof. If $(\alpha, \beta) \in F_1$ and $\alpha \neq 0$, then by (45) $b_3 < 0$ and $b_0 > 0$, and

$$B(h_1) = -5(\lambda-1)^2(\lambda+1)^2(2\lambda+1)^2(\alpha-\beta+1)^2\psi_1(\alpha, \beta, \lambda), \tag{52}$$

where $h_1 < 0$ is given in (4), and $\psi_1(\alpha, \beta, \lambda) = 36(\lambda+2)\alpha + (30\lambda^2 + 57\lambda + 21)\beta + 2(2\lambda+1)(7\lambda^2 + 10\lambda + 1) > 0$ for $(\alpha, \beta) \in F_1$, since the straight line $L_1 = \{(\alpha, \beta) \mid \psi_1(\alpha, \beta, \lambda) = 0\}$ crosses L_b and $\tilde{L}$ at the same point $(-\frac{(\lambda+2)(2\lambda+1)}{9}, \frac{2(1-\lambda)}{3})$ and leaves F_1 entirely to its right side, see Fig. 19, hence $B(h_1) \leqslant 0$ by (52). This implies that $B(h)$ has exactly one positive zero. If $(\alpha, \beta) \in F_1$ and $\alpha = 0$, then it is not difficult to show that $B(h)$ has at most one positive zero.

If $(\alpha, \beta) \in \tilde{L}$, then $B(h) = 0$ becomes

$$-48(18\alpha + \lambda^2 + 7\lambda + 1)h^2 + \tilde{b}_1 h + 3\lambda^2(\lambda+2)(2\lambda+1)\alpha^2 = 0,$$

where $\tilde{b}_1$ is a polynomial in α and λ. Note that $\tilde{\alpha} = -\frac{1}{18}(\lambda^2 + 7\lambda + 1)$ is just the α-coordinate of $\tilde{L} \cap L_a$. We immediately obtain that $B(h)$ has at most one positive zero.

If $(\alpha, \beta) \in F_2$, then by (45) $b_3 > 0$ and $b_0 \geqslant 0$, so that the conclusion is straightforward. $\square$

Lemma 5.8. *If $\alpha < 0$, $\beta \leqslant 0$ and (α, β) is below L_0, then $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) \leqslant 3$ for any $\lambda \in (0,1)$.*

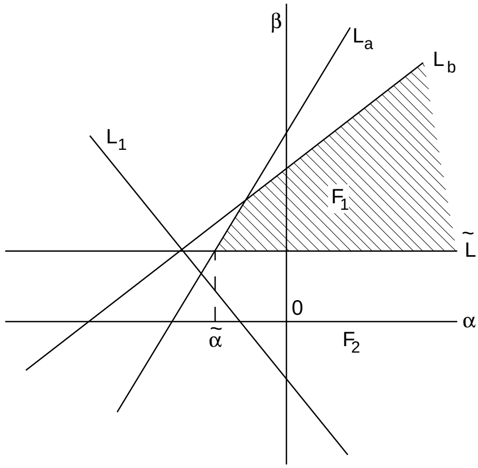

Fig. 19. The relative position of L_1 and F_1.

Proof. By Lemma 5.3 $L_0 \subset R_+ \cup G$, hence under the conditions of this lemma there are two cases:

(1) $(\alpha, \beta) \in F_2 \cap \{\alpha < 0, \beta \leqslant 0\} = E_2$;
(2) $(\alpha, \beta) \in R_+ \cap \{\alpha + \beta P(0) + Q(0) > 0, \alpha < 0, \beta \leqslant 0\} = E_3$.

We first consider case (1). Note that $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) \leqslant N(C_P \cap C_R) + 1$, let us show that $N(C_P \cap C_R) \leqslant 2$ for $(\alpha, \beta) \in E_2$. We are going to use the following properties that hold for this case:

(A) $b_3 > 0$ implies that the vector field (8) is upwards with respect to C_R for $h \gg 1$.
(B) By (46) $\frac{x_2}{y_2} - \frac{\lambda - 1}{3} > 0$, hence C_P stays below C_R for $h \gg 1$.
(C) By Lemma 5.7 there are at most two "tangent points" on C_R with respect to the vector field (8).
(D) By (49) $P(0) < R(0)$.
(E) By (42) $y_2 > 0$ and $y_0 > 0$, hence $(0, P(0))$ is located in the left or the right strip of (c_i) in Fig. 14, if $Y(h) = 0$ has two positive roots.

By property (B), C_P intersects C_R at most at 2 branches; as such $N(C_P \cap C_R) \leqslant 2$ for case (c4) of Fig. 14, since all of its branches are monotonically decreasing and $P'(h) > 0$. If C_P intersects the most right branch of C_R, then by property (A) there is always a tangent point on C_R right to the most right intersection point of C_P and C_R. Together with property (B), this implies that C_P cannot meet both the middle and the right branches of C_R in case (c1); otherwise two more tangent point appear, contradicting property (C). If C_P cuts one branch of C_R in case (c1) and $N(C_P \cap C_R) > 2$, then this number is at least 4, also contradicting the property (C). In cases (c2) and (c3), C_P must meet the most right branch of C_R, and may cut one more of the other branches. If C_P intersects one branch of C_R more than once then it cuts C_R at least 3 times, and this causes at least two more tangent points on C_R, also contradicting property (C). The analysis for cases (a_i) and (b_i) is the same, and is simpler than for case (c_i). The proof for $(\alpha, \beta) \in E_2$ is hence finished. We next consider case (2). We will mainly study $N(C_\omega \cap C_U^{(1)})$ in (h, ω)-plane. By Lemma 5.2 $E_3 \subset \{(\alpha, \beta) \mid \alpha - \beta + 1 > 0\}$, hence by using (26) and (4) we have

$$U(h_1) + 1 = \frac{(\lambda + 2)(\alpha - \beta + 1)}{-(\lambda + 1)\beta + \lambda + 3} > 0.$$

This means that in (h, ω)-plane $C_U^{(2)}$ crosses the line $\{h = h_1\}$ between the saddle point S_1 and the node $N_1(h_1, -1)$ (see Fig. 8). In other words; $A(h_1) > 0$. $a_3 > 0$ implies $A(-\infty) < 0$, hence there exists a $\tilde{h} < h_1 < 0$ such that $A(\tilde{h}) = 0$.

From (30) it is easy to see that for $\alpha < 0$, $a_0 = 0$ is given by $\psi_2(\alpha, \beta, \lambda) = (10\lambda^2 - 11\lambda + 10)\alpha + 12\lambda(1 - \lambda)\beta + 12\lambda^2 = 0$. Denote by L_2 the straight line $\{(\alpha, \beta) \mid \psi_2(\alpha, \beta, \lambda) = 0\}$, then $L_2 \cap L_a = \{(\tilde{\alpha}, \tilde{\beta})\}$, where

$$\tilde{\alpha} = -\frac{\lambda(\lambda^2 + \lambda + 1)}{(\lambda + 2)(2\lambda + 1)}, \quad \tilde{\beta} = \frac{10\lambda^4 - 13\lambda^3 - 21\lambda^2 - 13\lambda + 10}{6(1 - \lambda)(\lambda + 2)(2\lambda + 1)}. \tag{53}$$

For $\lambda \in (0, 1)$, $\tilde{\beta} = 0$ has a unique root at $\tilde{\lambda} \approx 0.425$. If $\lambda \in (0, \tilde{\lambda}]$, then $E_3 \subset \{(\alpha, \beta) \mid a_3 > 0, a_0 \leqslant 0\}$. This fact, together with $A(\tilde{h}) = 0$, $\tilde{h} < 0$, implies $C_\omega \cap C_U^{(1)} = \emptyset$. In fact, $a_3 > 0$ means $A(+\infty) > 0$, hence $C_\omega \cap C_U^{(1)} \neq \emptyset$ needs at least two positive zeros of $A(h)$.

If $\lambda \in (\tilde{\lambda}, 1)$, we need to divide E_3 into three parts by L_2 and L_1, where L_1 is the same as in Fig. 19. Thus $E_3 = \bigcup_{i=1}^{3} E_{3i}$, where

$$E_{31} = \{(\alpha, \beta) \in E_3 \mid \beta \leqslant 0, \psi_1(\alpha, \beta, \lambda) > 0\},$$

$$E_{32} = \{(\alpha, \beta) \in E_3 \mid \psi_1(\alpha, \beta, \lambda) \leqslant 0, \psi_2(\alpha, \beta, \lambda) \geqslant 0\},$$

$$E_{33} = \{(\alpha, \beta) \in E_3 \mid \psi_2(\alpha, \beta, \lambda) < 0\}.$$

The discussion for $(\alpha, \beta) \in E_{33}$ is the same as above; the situation for $(\alpha, \beta) \in E_{31}$ is the same as for $(\alpha, \beta) \in F_1$ in the proof of Lemma 5.7, observing that $b_3 < 0$ and $b_0 > 0$, and E_{31} is located right to L_1. Hence $B(h) = 0$ has only one positive root, implying $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) \leqslant 3$ (Lemma 5.4(iii)). To finish the study for $(\alpha, \beta) \in E_{32}$, we also need to consider $L_3 = \{(\alpha, \beta) \mid \psi_3(\alpha, \beta, \lambda) = 0\}$, where $\psi_3 = \alpha + \lambda\beta + \lambda^2$. Calculation shows that L_1 is strictly between L_2 and L_3 for $\lambda > \tilde{\lambda}$ and $(\alpha, \beta) \in E_3$, see Fig. 20.

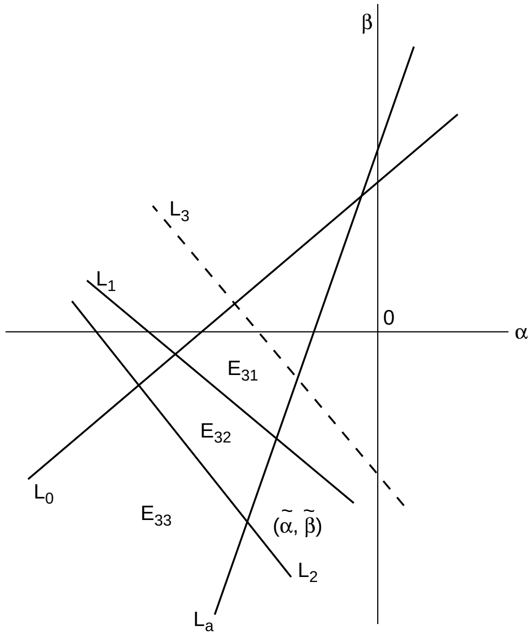

Fig. 20. The relative positions of L_1, L_2 and L_3.

Note that

$$\tilde{\beta} + \frac{\lambda}{1-\lambda} = \frac{(\lambda^2+\lambda+1)(10\lambda^2-11\lambda+10)}{6(1-\lambda)(\lambda+2)(2\lambda+1)} > 0,$$

hence for $(\alpha, \beta) \in E_{32}$ we have

$$(1-\lambda)\beta + \lambda > 0. \tag{54}$$

By (26), (18) and (4), for $(\alpha, \beta) \in E_{32}$

$$U(0) - \omega_0 = \frac{(\lambda-1)\psi_2(\alpha, \beta, \lambda)}{(10\lambda^2-11\lambda+10)((1-\lambda)\beta+\lambda)} \leqslant 0, \tag{55}$$

the equality holds only for $(\alpha, \beta) \in L_2$, and

$$U(h_2) - \lambda = \frac{-(2\lambda+1)\psi_3(\alpha, \beta, \lambda)}{(\lambda+2)\beta+3\lambda^2+\lambda}. \tag{56}$$

For $(\alpha,\beta)\in E_{32}$ we have $a_3>0$ and $a_0\geqslant 0$ (the equality holds only for $(\alpha,\beta)\in L_2$). Let us show that $A(h)$ has no positive zero for $(\alpha,\beta)\in E_{32}$, implying $C_\omega\cap C_U^{(1)}=\emptyset$. In fact, we have found a negative zero $\tilde h$ of $A(h)$, so it is enough to find one more negative or zero root of $A(h)=0$. If $U(0)-\omega_0=0$, then $C_U^{(1)}$ passes through the saddle point S_3 in Fig. 8, this gives the zero root of $A(h)=0$. If $U(0)-\omega_0<0$, then $C_U^{(1)}$ cuts the line $\{h=0\}$ between the singularities O and S_3. L_3 is above E_{32}, hence $\psi_3(\alpha,\beta,\lambda)<0$ for $(\alpha,\beta)\in E_{32}$. By (23) and (4) it is easy to see that $(\lambda+2)\beta+3\lambda^2+\lambda$ is positive, zero, or negative corresponding to, respectively, $\bar h<h_2$, $\bar h=h_2$ or $\bar h>h_2$. Eq. (54) implies $\bar h<0$. It is clear by using this fact and (56), that in any case $C_U^{(1)}$ must intersect the unstable manifold going from the saddle point S_3 to the node N_2 in Fig. 8, and this gives one more negative root of $A(h)=0$. $\square$

Theorem 5.9.

(1) *For any $\lambda\in(0,1)$, $\Sigma_\lambda^{(3)}$ has a unique minimum at some point M_λ and to the right of M_λ (including the point), $\Sigma_\lambda^{(3)}$ has no inflection point nor higher tangency point.*
(2) *For $\lambda\in(0,\lambda^*]$, $\Sigma_\lambda^{(3)}$ has a unique inflection point to the left of M_λ.*
(3) *For $\lambda\in(\lambda^*,1)$ and to the left of M_λ, $\Sigma_\lambda^{(3)}$ has only the following three possibilities*:
   (i) *there are exactly two inflection points*;
   (ii) *there is a unique quadruple point*;
   (iii) *there is no inflection point nor higher tangency point.*

   In the above statement, "inflection point" means a point on $\Sigma_\lambda^{(3)}$ with tangency of order 3.

Moreover, for $0<\lambda-\lambda^*\ll 1$ case (i) *occurs, for $0<1-\lambda\ll 1$ case* (iii) *occurs, and at some $\lambda^\circ\in(\lambda^*,1)$ case* (ii) *occurs.*

Proof.

(1) By Lemmas 3.1 and 3.3, $\Sigma_\lambda^{(3)}$ has at least one minimum; we suppose the first one to occur at M_λ. Since $Q(h)>0$ for all h, the tangent line $\mathscr{L}_{\alpha,\beta}$ at M_λ satisfies $\beta=0$ and $\alpha<0$. The straight line in (P,Q)-plane, passing through the two points $(P(0),Q(0))$ and $(0,-\alpha)$ is above $\mathscr{L}_{\alpha,0}$, for $P<0$, this implies that in (α,β)-plane the point $(\alpha,0)$ is below L_0. The point (α_T,β_T) stays below L_0 for $(\alpha_T,\beta_T)\in C_\lambda$, at least up to the value of h, corresponding to the first inflection point on $\Sigma_\lambda^{(3)}$ to the right of M_λ. Thus the conclusion (1) follows by Lemma 5.8. In fact, if M_λ is not a global minimum of $\Sigma_\lambda^{(3)}$, or if there is a inflection point right to M_λ, we would find a $\mathscr{L}_{\alpha,\beta}$ with $\alpha<0$ and $\beta\leqslant 0$, such that $N(\mathscr{L}_{\alpha,\beta}\cap\Sigma_\lambda^{(3)})\geqslant 4$, see Fig. 21(a) and (b), respectively. Here we need to use Lemma 3.1 again. By the same reason, M_λ is a point on $\Sigma_\lambda^{(3)}$ with tangency of order 2.

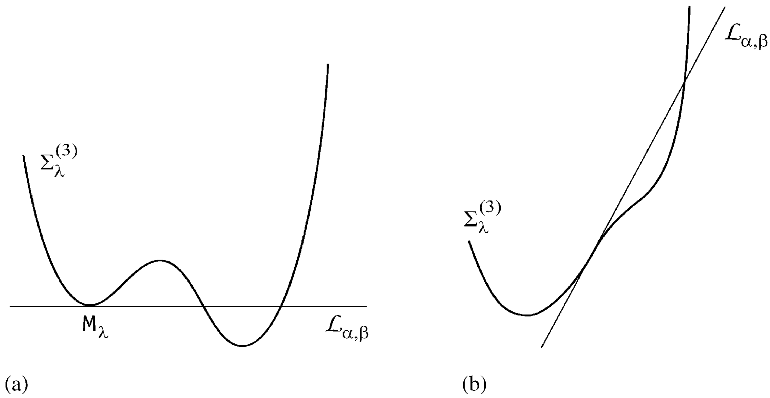

Fig. 21. Hypothetical shapes of $\Sigma_\lambda^{(3)}$ right to M_λ.

(2) For $\lambda \in (0, \lambda^*]$ by Theorem 3.6 and Remark 3.7 it is necessary to have at least one inflection point on $\Sigma_\lambda^{(3)}$ to the left of M_λ, and by Lemma 4.5 the first inflection point occurs above L_0. Since $L_0 \subset R_+ \cup G$ (Lemma 5.3), by Lemmas 4.6 and 5.6, this inflection point is unique for $(\alpha, \beta) \in R_+ \cup G$. We claim that it is also unique for $(\alpha, \beta) \in \mathbb{R}^2 \backslash (R_+ \cup G) = F_1 \cup \tilde{L} \cup F_2$. In fact, if there are more inflection points, then at least two more of them occur left to M_λ. We have proved in [DL2] that $\Sigma_\lambda^{(3)}$ has a unique inflection point for $\lambda = 0$, hence by varying λ, we will find a $\lambda \in (0, \lambda^*]$ such that $\Sigma_\lambda^{(3)}$ has a point of tangency of order at least 4, i.e. C_P and C_R have a tangent point that is at least triple. By Lemma 5.7, this is obviously impossible for $(\alpha, \beta) \in F_1 \cup \tilde{L}$. If $(\alpha, \beta) \in F_2$, then as we discussed for $(\alpha, \beta) \in E_2 \subset F_2$ in the proof of Lemma 5.8, in case (c1) of Fig. 14, C_P can intersect only one branch of C_R, and $N(C_P \cap C_R)$ is even, hence the existence of a third order tangency of C_P and C_R gives at least 3 "tangent points" on C_R with respect to the vector field (8), contradicting Lemma 5.7. In cases (c2) and (c3) C_P must intersect the most right branch, and there is a tangent point on C_R right to the most right intersection point of C_P and C_R. This fact, together with Lemma 5.7, gives no possibility to have a third-order tangency of C_P and C_R. The conclusion for case (c4) is obviously true. The discussion for cases (a_i) and (b_i) is the same.

(3) We have proved that when $\lambda \in (0, \lambda^*]$ there is an inflection point on $\Sigma_\lambda^{(3)}$, left to M_λ. When λ increases and passes the value λ^*, then by Theorem 3.6, $\Sigma_\lambda^{(3)}$ changes its convexity near its left end point $(P(0), Q(0))$. Hence for $0 < \lambda - \lambda^* \ll 1$ one more inflection point on $\Sigma_\lambda^{(3)}$ must occur. In any case, if for $\lambda \in (\lambda^*, 1)$, $\Sigma_\lambda^{(3)}$ has two inflection points to the left of M_λ and the corresponding values (α, β) stay in $R_+ \cup G$, then by Lemmas 4.6 and 5.6 there are no more inflection point nor higher tangency points on $\Sigma_\lambda^{(3)}$. The last conclusion is still valid for

$(\alpha, \beta) \in \mathbb{R}^2 \backslash (R_+ \cup G)$ by using the same arguments as we did for case (2) above; under the condition that for some $\lambda \in (\lambda^*, 1)$ (for example $\lambda \sim 1$) $\Sigma_\lambda^{(3)}$ has no inflection point nor higher tangency point for $(\alpha, \beta) \in \mathbb{R}^2 \backslash (R_+ \cup G)$. This is a quite easy observation.

By calculations, from (30) and (45), one can check that for $\lambda > 0.5$ the case (iii) occurs. We will not carry it out, since $\lambda = 1$ corresponds to the symmetric case, and as we mentioned in Section 2, it is easy to prove that for $\lambda \sim 1$ the case (iii) occurs.

Finally, in between cases (i) and (iii), case (ii) must happen for some $\lambda \in (\lambda^*, 1)$. The fact that there is no quadruple (nor higher tangency) point on $\Sigma_\lambda^{(3)}$ for $(\alpha, \beta) \in \mathbb{R}^2 \backslash (R_+ \cup G)$ was mentioned above several times. $\square$

Remark. We believe, but are not able to prove, that case (ii) occurs at unique value λ° implying that we have case (i) (resp. case (iii)) to the left (resp. to the right) of λ°.

6. The proof of Theorem 1.1

6.1. *Proof of statement (1)*

For any $\lambda \in (0, 1)$ and any constants α and β, if a straight line $\mathscr{L}_{\alpha,\beta}$ cuts $\Sigma_\lambda^{(3)}$ at least twice, taking into account the multiplicity, then we may move it to a new position $\mathscr{L}_T = \mathscr{L}_{\alpha_T, \beta_T}$, such that $\mathscr{L}_T$ is a tangent line to $\Sigma_\lambda^{(3)}$, having the same intersection number with $\Sigma_\lambda^{(3)}$. Hence, by Theorem 5.9, the sharp upper bound of the intersection points is 4.

By Theorem 3.5, any straight line cuts $\Sigma_\lambda^{(1)}$ or $\Sigma_\lambda^{(2)}$ at most twice, taking the multiplicity into account. Since $Q(h) > 0$ for all h and by Lemmas 3.3 and 3.5 (see Fig. 7) it is clear that a straight line cutting $\Sigma_\lambda^{(1)}$ twice cannot cut $\Sigma_\lambda^{(2)}$, and a straight line cutting $\Sigma_\lambda^{(2)}$ twice cannot cut $\Sigma_\lambda^{(1)} \cup \Sigma_\lambda^{(3)}$, implying $n_1 + n_2 \leqslant 2$.

Hence, to finish the proof of Statement (1), we only need to show that if $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(1)}) = 2$, then $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) \leqslant 3$. Actually, we will prove that in this case $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) \leqslant 2$. In fact, by Lemmas 3.1–3.3 and Theorem 3.5, the line $\mathscr{L}_{\alpha,\beta}$ must cut the straight line L (see (9) and Fig. 7) at a point, left to $M_1(P_1(0), Q_1(0))$, hence it cuts the Q-axis below the origin. If $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) = 4$ (this happens only for $\lambda > \lambda^*$, see Theorem 5.9 and Fig. 7(b)), then by Lemma 3.3 and Theorem 3.6 $\mathscr{L}_{\alpha,\beta}$ must cut the line L at a point left to M_3 and cut Q-axis above the origin, giving a contradiction; if $N(\mathscr{L}_{\alpha,\beta} \cap \Sigma_\lambda^{(3)}) = 3$ (this may happen both for $\lambda \in (0, \lambda^*]$ and $\lambda > \lambda^*$, see Figs. 7(a) and (b)), then $\mathscr{L}_{\alpha,\beta}$ must cut the line L at a point right to $M_3(P_3(0), Q_3(0))$, also leading to a contradiction. $\square$

6.2. Proof of statements (2)–(5)

For a straight line $\mathscr{L}_{\alpha,\beta}$, let $N(\mathscr{L}_{\alpha,\beta}\cap\Sigma_\lambda^{(1)}) = i$, $N(\mathscr{L}_{\alpha,\beta}\cap\Sigma_\lambda^{(2)}) = j$ and $N(\mathscr{L}_{\alpha,\beta}\cap\Sigma_\lambda^{(3)}) = k$. The above analysis shows that if $k = 3$ or 4 then $i \leqslant 1$ and $j = 0$. By a similar analysis it is easy to see that if $k = 0$ then $i + j \leqslant 2$ and $\min(i,j) = 0$; if $k = 1$ then $i \leqslant 1$ and $j \leqslant 1$; if $k = 2$ then $(i,j) = (2,0), (1,0)$ or $(0,1)$.

For $\lambda \sim 0$, $\Sigma_\lambda^{(1)} \cup \Sigma_\lambda^{(3)}$ is close to the curve $\Sigma_- \cup \Sigma_+$ of the cuspidal loop case, see Fig. 8 of [DL2], hence the configurations $((2,0),2)$ and $((1,0),3)$ are realizable. Finally, let us show that the configuration $((1,0),4)$ is also realizable. By Theorem 5.9, if $0 < \lambda - \lambda^* \ll 1$, then $\Sigma_\lambda^{(3)}$ has two inflection points, left to the minimum point. Denote by $-\beta_1$ the slope of the tangent line of $\Sigma_\lambda^{(3)}$ at its left inflection point, then we can find straight lines $\mathscr{L}_{\alpha,\beta}$ with $\beta \sim \beta_1$ such that $N(\mathscr{L}_{\alpha,\beta}\cap\Sigma_\lambda^{(3)}) = 4$. From Lemma 3.4 and the discussion in the proof of Lemma 4.6 and Theorem 5.9, the left inflection point of $\Sigma_\lambda^{(3)}$ tends to its left end point $(P_3(0), Q_3(0))$ as $\lambda \to \lambda^*$. By Lemmas 3.3 and 2.2 we know that the slope of the tangent line $\mathscr{L}^*$ of $\Sigma_\lambda^{(3)}$ at its left end is

$$\left.\frac{Q_3(0)}{P_3(0)}\right|_{\lambda=\lambda^*} = \left.\frac{I_2^{(3)}(0)}{I_1^{(3)}(0)}\right|_{\lambda=\lambda^*} \approx -0.935.$$

Note that the left end point of $\Sigma_\lambda^{(1)}$ is $(-1,1)$, hence $\mathscr{L}^*$ must cut $\Sigma_{\lambda^*}^{(1)}$ transversally, proving the claim. $\square$

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